THE DEVELOPMENT OF TRIGONOMETRY FROM REGIOMONTANUS TO PITISCUS

SISTER MARY CLAUDIA ZELLER
of the Sisters of St. Francis
of Mary Immaculate
Joliet, Illinois

A DISSERTATION
SUBMITTED IN PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR THE
DEGREE OF DOCTOR OF PHILOSOPHY
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1944

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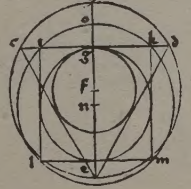

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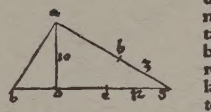
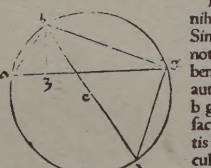


duæ differentiæ sint datæ, & ipsa perpendicularis a d data. Dico, q; omnia latera trianguli nota concludentur. Per artem rei & census hoc problema absoluemus. Detur ergo differentia laterum ut 3, differentia casuum 12. & perpendicularis 10. pono pro basi unã rem, & pro aggrega to laterum 4 res, nã proportio basis ad congerem laterũ est ut h g ad g e, scilicet unius ad 4. erit ergo b d $\frac{1}{4}$ rei, minus 6, sed a b erat 2 res demptis $\frac{1}{4}$. duco a b in se, producuntur 4 census & 2 $\frac{1}{4}$. demptis 6 rebus. itẽ b d in se facit $\frac{1}{16}$ census, & 36 minus 6 rebus: huic addo quadratum de 10 qui est 100. colliguntur $\frac{1}{16}$ census & 136 minus 6 rebus. æquales uidelicet 4 censibus & 2 $\frac{1}{4}$ demptis 6 rebus. Restaurando itaq; defectus, & auferendo utrobiq; æqualia, quemadmodum ars ipsa præcipit, habebimus census aliquot æquales numero, unde cognitio rei patebit, & inde tria latera trianguli more suo innotescẽt.

XXIIII.

Datis tribus lateribus trianguli rectilinei, diametrum circuli eum circumscribentis inuenire.



Hæc tametsi de angulis trianguli inueniendis nihil proponat, utilis tamen sequentibus uidebit. Sint tria latera a b, b g & g a, trianguli a b g nota, quærimus diametrum circuli eum circũscribentis. Esto circulus huiusmodi a b g d, oportet autem duos angulos quicunq; fuerint trianguli a b g esse acutos, qui sint, uerbi gratia, a & g, quos facile cognosces, si primo triangulorum libello satis incubuisti: demittatur q; à puncto b perpendicularis b 3, & diameter circuli prædicti b e d, cuius terminus d copuletur puncto g per lineã d g. habes itaq; duos triangulos a b 3 & b d g æquiangulos, uterq; enim angulorum b a g & b d g in circumferentia consistens susc ipit arcum b g, sed uterq; angulorum a 3 b & b g d rectus est: a 3 b quidem ex dispositione figuræ, b g d autem ex dispositione & 30. tertij elementorum, quare & tertius tertio æqualis cõ uincetur: unde & per quartã sexti proportio 3 b ad b g est ut a b ad b d. tres autem harum notæ sunt, duæ scilicet a b ad b g per hypothesim: perpendicularis uero b 3 ex primi huius inuenitur, ergo & quarta, quæ est diameter circuli, nota ueniet, quod expectabas ostendendum. Elegimus autem duos angulos acutos, ut perpendicularis intra triangulum coartaretur facilitatis gratia, nam si caderet extra, parumper uariandus esset processus. Quòd si acciderit quadratum alicuius trium laterum quadratis duorum reliquorum laterum simul iunctis æquiualere: uerbi gratia, quadratum a g æquari duobus quadratis linearum a b & b g, erit a g diameter circuli circumscribentis triangulũ, nec; ampliori quæsito opus est: fiet enim per ultimã primi elementorum angulus a b g rectus, & ideo per cõuersionem 30. tertij a b g semicirculus habebitur, & a g diameter circuli, quod erat exequendum.

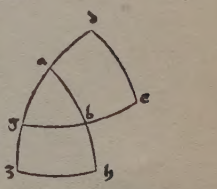
XXV.

Si basim trianguli notam acceperimus cũ perpendiculari siue area trianguli, cum quoq; quem basis subtendit angulum datum habuerimus, utri

ri ca præter polum circuli cuiuslibet, extra tamen circumferentiam eius signatũ, geminos demittere liceat arcus perpendiculares ad circumferentiam ipsius circuli. possibile est enim per punctum signati & polum circuli transire circuli alium magnum tertij huius docente. circuli igitur hoc pacto descripti, duo arcus inter punctũ signatum & circumferentiam circuli sacentis intercepti, perpendiculares erunt ex huius ad circumferentiam circuli sacentis. In figuratione itaq; præsenti a puncto b prodibit perpendicularis, unus ad partem inclinatiõis, reliquus autem ad partem oppositam. Idem quoq; accidet cæteris punctis in utraq; circumferentiarũ signatis, uerum hoc non interturbabit syllogismũ nostrum, nam hij duo perpendiculares cum sint æquales semicircumferentiæ per tertij huius, communis animi conceptio eundem ipsius communem donabit finum. quicquid igitur de uno arcuũ prædicabit theorema nostrum, & de reliquo demonstratũ habebit.

XVI.

In omni triangulo rectangulo omnium laterum sinus ad sinus angulorum, quos subtendunt, eadem est proportio.



Sit triangulus a b g angulum b rectum habẽs. Dico q; proportio sinus lateris a b ad sinum anguli a g b eadem est proportioni sinus lateris b g ad sinũ anguli b a g, & proportioni, quam habet sinus lateris a g ad sinum anguli a b g, quod sic demonstrabimus. Necesse est, aut utrunq; angulorũ a & g esse rectum, aut alterũ eorũ duntaxat, aut nullum. Si uterq; eorum rectus est, erit per hypothesim & huius pũctus a polus circuli b g, b autem polus circuli a g & g polus circuli a b, quare per diffinitionem unusquisq; trium arcuũ dictorũ determinabit quantitatem anguli sibi respicientis. idem igitur erit sinus cuiuslibet trium laterũ & anguli sibi oppositi, & ideo sinus omniũ laterũ ad sinus angulorũ se respicientiũ, proportionem eandem accipiunt uidelicet æqualitatis. Si autem alter duntaxat angulorũ a & g fuerit rectus, sit ille, uerbi gratia, angulus g. tradidit autem & hypothesis anguli b rectum, quare per huius est polus circuli b g, & per huius uterq; arcuũ b a, & a g quadrans circumferentiæ magnæ. per diffinitionem igitur unusquisq; arcuũ a b, b g & g a quantitatem anguli se respicientis determinabit, eritq; idem sinus cuiuslibet laterum & anguli se respicientis conuertendo diffinitionem sinus anguli. unde postremo constabit omnium laterum sinus ad sinus angulorũ se respicientiũ, eandem habere proportionem scilicet æqualitatis. Qd' si neuter angulorũ a & g rectus offeratur, non poterit aliquis trium arcuũ lateralium esse quadrans circumferentiæ, quemadmodũ ex huius trahitur, uerum in triplici uarietate habebũtur. Si enim uterq; angulorũ a & g fuerit acutus, erit per huius uterq; arcuũ a b & b g minor quadrante, unde & per huius arcus a g minor quarta circumferentiæ. Crescat igitur arcus g a uersus a donec fiet quadrans a d, secundum cuius cordam quæ est costa quadrati magni super punctũ a facto polo describatur circulus magnus, secans arcum g b prolongatũ in puncto e. prolongentur demum arcus a g ad partem puncti g, donec quadrans habebitur a 3, cuius corda super polo a circumducta pariat circuliũ occurrentem arcui a b continuato in puncto h. Piscinamus hoc pacto unam figurationem. Si uero uterq; angulorum a & g obtusum se præbeat

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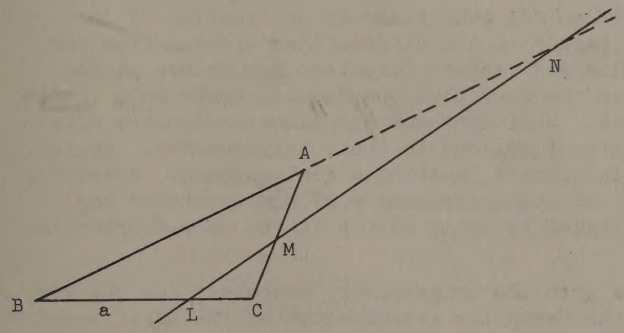
PREFACE

Trigonometry, as a science of solving problems in astronomy, required as a foundation the preparation of a table of chords in a circle corresponding to different angles. Computations, involving the use of the Pythagorean theorem, were made for such tables by the Babylonians about fifteen hundred years before the Christian Era. In Greece tables appear to have been prepared based on the Babylonian work first by the "Father of Greek Astronomy," Hipparchus, in the second century B.C. These tables using 120 as a diameter and those of Menelaus of about 100 A.D. have disappeared, as well as treatises "On chords in a circle" listed in ancient records as having been written by both of these men.

Hipparchus used a method of projection to enable him to make a beginning in spherical trigonometry. However, it remained for Menelaus, to give a foundation for a real trigonometry of the right spherical triangle. The "theorem of Menelaus" states that if in any plane triangle ABC a straight line cuts the sides a, b, and c in three points L, M, and N on the three sides, one or all extended; then the product of the first three segments of each of the sides BL, CM, and AN, taken in order around the triangle, equals numerically the product of the second three segments LC, MA, and NB. In modern geometry this would be written $\frac{AN}{NB}\frac{BL}{LC}\frac{CM}{MA} = -1$.



However, the Greeks did not use negative lines in their geometry and wrote this relation as the equality of the ratio $\frac{AN}{NB}$ as equal to the product of the two ratios $\frac{LC}{BL}$ and $\frac{MA}{CM}$. In spherical geometry a triangle ABC is formed by great circle arcs on a sphere, and is cut across by a fourth great circle arc in points L, M, and N. Then the theorem of Menelaus states here the equality of the following ratios, $\frac{\text{Chord } (2AN)}{\text{Chord } (2NB)} = \frac{\text{Chord } (2LC)}{\text{Chord } (2BL)} \cdot \frac{\text{Chord } (2MA)}{\text{Chord } (2CM)}$.

This was the form of the theorem of Menelaus which became the foundation of the Greek trigonometry of chords, more completely developed by Ptolemy (c. 150 A.D.) and finally of the trigonometry of sines, cosines, tangents and cotangents developed by the Moslems.

By applying the theorem of Menelaus to right spherical triangles Ptolemy was able to develop a somewhat systematic spherical trigonometry for astronomy which sufficed for Europeans largely up to the time of printing.

The Greeks used only chords of angles and of their supplements. The connection with the trigonometry of sines and cosines, first developed by the Hindus, is indicated by the relationship,

$$\frac{\text{Chd. } (2\alpha)}{2R} = \sin\alpha;\ 5\ \frac{\text{Chord } (180^\circ - 2\alpha)}{2R} = \sin(90^\circ - \alpha)$$

or $\cos \alpha$. The hindus used cotangents in shadow problems, but the consistent use of the cotangent and tangent as a trigonometric function was introduced by Arabic speaking mathematicians.

The spherical trigonometry of the right spherical triangle with some ten formulas is sufficiently complicated to make the astronomical problems quite difficult of of solution. The Arabs had a better appreciation of the generality of the theorem of Menelaus than did the Greeks. This combined with replacing the chords by sines and cosines enabled the Arabs as early as 1025 A.D. to formulate the sine law of plane trigonometry and the same for spherical trigonometry. Not long thereafter the introduction by capable astronomers of the tangent and cotangent functions with tables for the same further modernized the Arabic trigonometry.

The systematization of plane as well as spherical trigonometry was finally achieved by the great Persian mathematician, Nasir ad-Din at-Tūsī who lived in Tus in the second half of the thirteenth century.

The Moslem mathematics began to arrive in Europe first in the eleventh and twelfth centuries, largely in Latin and Hebrew translations from the Arabic made in Spain. The Sphaerics of Menelaus was first found in both Latin and Hebrew translations from the Arabic, before it was found in the Greek. Unfortunately the work of the Eastern Arab Nasir ad-Dīn did not reach Europe through Spain. In the fifteenth century Regiomontanus, (1436-1476), an able mathematician, prepared a treatise on plane and spherical trigonometry, without using the tangent function but employing the sine law both plane and spherical. This trigonometry was prepared before Regiomontanus learned the value of the tangent function of which a table appears in one of his later works. After the death of Regiomontanus in 1476, the manuscript of his trigonometry was not made available until 1533 when the work was published.

The spherical trigonometry continued for many years to be troubled by the necessity of a multitude of formulas due partly to the difficulties arising from the fact that the negative values of the cosine and tangent functions for obtuse angles were not recognized. Quite as serious was the fact that the Spanish Arabs made little use of the tangent and cotangent functions. Regiomontanus and almost a century after him Copernicus used only the sine and cosine functions in their trigonometry. Regiomontanus did apply the algebra in his trigonometry, marking a real advance. However the long delay of 57 years in publication of the trigonometry of Regiomontanus kept European trigonometry far behind that achieved by Nasir ad-Dīn in the late thirteenth century.

Rheticus who assisted Copernicus with the trigonometry necessary for the new theory of the sun as center of the universe began the introduction of the basic right triangle in plane trigonometry, with all six functions, and the extended applications to the spherical trigonometry.

Sister Claudia has in the present work given the detailed story of this transition, particularly from Regiomontanus through Copernicus and a host of other Europeans to Vieta and Pitiscus with a largely modern trigonometry.

Louis C. Karpinski

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INTRODUCTION

The early sixteenth century shows an increasing independence of thought and a growing spirit of scientific inquiry. The cultivation of pure mathematics and astronomy did away with much of the servile intellectual activity of medieval learning. For the historian of science, who sees the daring spirit of the past reflected in the scientific achievements of the present, this century is meaningful.

Prior to the Renaissance, trigonometry was the helpmate of astronomy. Naṣir ad-Dīn aṭ-Ṭūsī, an Arabic writer of the tenth century, was the first to produce a treatise on trigonometry independent of astronomy. This work, not known to Europeans until the fifteenth century, was, as is now commonly believed, the factor that influenced Regiomontanus to give the initiative in Europe to the development of trigonometry as an independent subject. The impetus given by Regiomontanus, fostered by Copernicus, furthered by Rheticus, and completed by Pitiscus and his contemporaries, had by 1600 freed trigonometry from its servitude to astronomy. The present work is offered as a contribution to the history of that emancipation.

The purpose of this thesis is to depict the development of trigonometry from Regiomontanus to Pitiscus. This period was covered over forty years ago by von Braunmühl in his Vorlesungen über Geschichte der Trigonometrie, which served as a basis for Tropfke's fifth volume (trigonometry) of his Geschichte der Elementar-Mathematik, published in 1923. The present study aims (1) to emphasize the part played by Regiomontanus, Copernicus, and Rheticus in the development of trigonometry in the sixteenth century, considering in a special way the dependence of Copernicus upon methods of Regiomontanus; (2) to summarize the enlarged modern conception of the part of Arabic trigonometry in the development of European trigonometry; and (3) to indicate how trigonometry evolved into a school subject through the advances made in its development by the European writers on trigonometry in the second half of the sixteenth century, culminating in the work of Pitiscus, which is essentially a modern exposition of trigonometry.

A ready access to most of the editions of the authors under consideration, afforded by the library of the University of Michigan, and to the bibliography of the trigonometric works of the sixteenth and seventeenth centuries, compiled by Professor L. C. Karpinski for Scripta Mathematica, greatly facilitated this study. Cornell University kindly lent and allowed photostats to be made of the separate trigonometric treatise of Copernicus, De lateribus et angulis triangulorum.

For the willing services of the librarians and especially for the kind encouragement of Professor L. C. Karpinski, I express appreciation and gratitude.

Chapter I

TRIGONOMETRY BEFORE REGIOMONTANUS

For a better understanding of the development of trigonometry during the sixteenth century, it is instructive to outline briefly its history up to the time of Regiomontanus. It was primarily for the purpose of perfecting astronomical calculations that trigonometry was cultivated at all for some 3000 years.[1] This explains "the rather startling fact that spherical trigonometry came to exist in a developed state earlier than plane trigonometry."[2] The cradle of civilization offers a starting place for this study.

1. BABYLONIANS

The chords of a circle as related to angles was given by the Babylonians, using the Pythagorean theorem. This has been indicated by the work of Neugebauer,[3] who gives their method of computing various chords:

> An historically important relation was employed for the reckoning of the lengths of a chord in a circle with respect to the perpendicular [from the center] which divides the chord into segments (this magnitude is sometimes called 'vers sine'). The formula for the calculation of the chord s follows from the diameter d and the perpendicular a; it is as follows:
>
> $$s = \sqrt{d^2 - (d - 2a)^2}.$$
>
> The corresponding inversion is:
>
> $$a = \tfrac{1}{2}\left(d - \sqrt{d^2 - s^2}\right).$$
>
> The correctness of this relation follows immediately from the figure, if one recognizes that the angle in a semicircle is a right angle, and uses the Pythagorean theorem. In all probability this theorem belongs to second millenium before Christ.

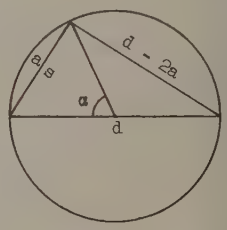

Fig. 1

An astrological calendar was determined for King Sargon I about 3000 B.C.[4] Instances such as these make the existence of trigonometric knowledge among the Babylonians very plausible, even if one is unable to say anything definite about its extent.[5]

2. EGYPTIANS

Numbers 56-60 of the Rhind Papyrus[6] reveal that the Egyptians possessed a knowledge of trigonometric ratios. In the translation and explanation of this papyrus by T. Eric Peet, Number 56 is representative:[7]

1. John David Bond, "The Development of Trigonometric Methods down to the Close of the XVth Century," Isis, 3:296, October, 1921.
2. Florian Cajori, A History of Mathematics (rev. ed., New York: Macmillan Co., 1926), p. 43.
3. Vorlesungen über Geschichte der Antiken Mathematischen Wissenschaften (Berlin: Springer, 1934), vol. I, p. 168.
4. Johann Tropfke, Geschichte der Elementar-Mathematik (2nd ed., Berlin: De Gruyter and Co., 1923), vol. V, p. 3.
5. Idem.
6. "The Rhind Papyrus as it now stands in the British Museum consists of two pieces separately mounted between sheets of plate glass and numbered 10057 and 10058 respectively. These two pieces once formed a single roll.... There is now a gap between them, and the Historical Society of New York possesses a number of fragments which come from this gap" (T. Eric Peet, The Rhind Mathematical Papyrus [London: Hodder and Stoughton Limited, 1923], p. 1).
7. Ibid., pp. 97-98.

Example of reckoning out a pyramid 360 in length of side and 250 in its vertical height. Let me know its batter. You are to take half of 360: it becomes 180. You are to reckon with 250 to find 180. Result 1/2 + 1/5 + 1/50 of a cubit. A cubit being 7 palms, you are to multiply by 7.

1	7
1/2	3 1/2
1/5	1 1/3 + 1/15
1/50	1/10 + 1/25

Its batter is 5 1/25 palms.

The meaning of this and the following examples depends on the interpretation given to the three terms whȝ -tbt, pr-m-wś and śkd. We can make the śkd correspond with this ratio if we take the pr-m-wś to be the vertical height DG, and the whȝ -tbt to be the side of the base, for the angle of slope (DF̂G) of a side QDR is one whose cotangent is half a side of the base divided by the vertical height, i.e., GF/GD.

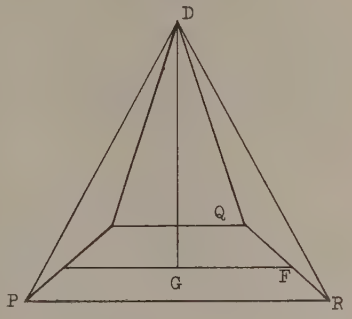

Fig. 2

3. GREEKS

The Greeks built their scientific structures upon the foundations erected by the Babylonians and Egyptians. The earliest Greek mathematicians and astronomers were travelers who acquired their knowledge from Babylonian and Egyptian priests.

THALES OF MILETUS (640-548 B.C.), who spent many years in Egypt, learned to determine the height of a pyramid from its shadow, and the distance of a ship from the harbor of Miletus from the similarity of triangles.[8]

ANAXIMANDER, a contemporary and pupil of Thales, introduced the sundial of the Babylonians into Greece. From the lengths of the sundial and its shadow, the determination of the height of the sun leads to the tangent of its angle of ascension.[9]

The only extant treatise of ARISTARCHUS OF SAMOS (ca. 310-230). On the Sizes and Distances of the Sun and Moon, is of great mathematical interest because it contains calculations of ratios which are in reality trigonometric ratios.[10] Proposition 12 from the translation and commentary by Heath is:[11]

> The diameter of the circle which divides the dark and the light portions in the moon is less than the diameter of the moon, but has to it a ratio greater than that which 89 has to 90.

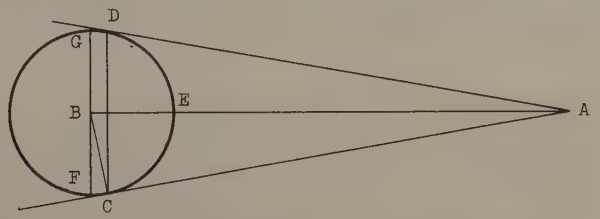

Fig. 3

8. A. von Braunmühl, Vorlesungen über Geschichte der Trigonometrie (Leipzig: Teuber, 1900), vol. I, p. 6.
9. Idem.
10. George Sarton, Introduction to the History of Science (Baltimore: William and Wilkins Co., 1927), vol. I, p. 156.
11. T. L. Heath, Aristarchus of Samos (Oxford: The Clarendon Press, 1913), p. 389.

This proposition is equivalent to proving that $1 > \cos 1^\circ > 89/90$[12] by using the relation $\cos \pi/2m = \sin (\pi/2 - \pi/2m) > (m - 1)/m$; $m > 1$.[13]

For let our eye be at A, and let B be the center of the moon when the cone comprehending both the sun and the moon has its vertex at our eye. Let AB be joined, and let a plane be carried through AB; this plane will cut the sphere, that is, the moon in a circle and the cone in straight lines. Let it cut the sphere in the circle DEC and the cone in the straight lines AD, AC, and CD. Therefore CD is a diameter of the circle which divides the dark and the bright portions in the moon. I say that CD is less than the diameter of the moon, but has to it a ratio greater than that which 89 has to 90.

For let FG be drawn through B parallel to CD, and let BC be joined. Then again, in the same way as before,[14] the angle DAC will be 1/45 th part of a right angle, and the angle BAC will be 1/90 th part of a right angle; but the angle BAC is equal to the angle CBF; therefore the angle CBF is also 1/90 th of the angle FBE; so that the circumference CF is also 1/90 th of the circumference FCE. Therefore the circumference CE has to the circumference ECF the ratio which 89 has to 90. Now DEC is double of CE, and GEF double ECF; therefore the circumference DEC has to the circumference GEF the ratio which 89 has to 90. And the straight line DC has to the straight line GF a ratio greater than that which the circumference DEC has to the circumference GEF. Therefore, also the straight line DC has to the straight line GF a ratio greater than that which 89 has to 90.

Assuming angle DAC = 2°, angle BAC = 1°.

cos BAC = AC/AB, but AC < AB so cos BAC < 1.

Angle CBF = angle BAC = 1°;
cos CBF = sin CBE.
sin CBE = sin (90° - CBF).

Aristarchus was the first to put forth the heliocentric hypothesis.

The "Theorem of the Broken Chord" gives evidence of the trigonometric knowledge of ARCHIMEDES (ca. 287-212). From Carl Schoy's[15] treatise, Tropfke[16] obtained his information on this interesting theorem of Archimedes.[17] The theorem and the first "Archimedean proof" given in the present work adhere closely to Tropfke's text.

"If in any circle two chords, AC and CD forming a broken line of unequal lengths, are drawn and from the midpoint B of the arc of the broken chord a perpendicular BH is drawn to the larger chord, then AH = BC + CD."

First Proof:

Make arc BZ = arc CB and HE = HC. Draw BC, BE, BA, BZ and ZA; then BH bisects CE, so CB = EB; also, arc CD = arc ZA. Angle B_1 is the inscribed angle measured by $\frac{1}{2}$ arc ZA. Angle A_2 is the

12. T. L. Heath, Op. cit., p. 335.
13. Ibid., p. 333.
14. In Proposition 4, Aristarchus proves that the circle which divides the dark and the bright portions in the moon is not perceptibly different from a great circle on the moon; he assumes angle CAD to be two degrees.
15. Die trigonometrischen Lehren des persischen Astronomen Abu'l Raihân Muh ibn Ahmd Al-Bîrûnî, dargestellt nach Al-Qânûn Al-Masûdî, edited after the death of the author by J. Ruska and H. Wieleitner (Hanover: H. Lefaire, 1927).
16. "Archimedes und die Trigonometrie," Archiv für Geschichte der Mathematik der Naturwissenschaften und der Technik, 10:433-435, Leipzig, 1928.
17. See also "Das Buch der Auffindung der Schnen im Kreis von Abu'l-Raihân Muh el-Bîrûnî." Übersetzt und mit Kommentar versehen von Henrich Suter in Zürich. Bibliotheca Mathematica, 9:11-80, Leipzig, 1911.

inscribed angle measured by $\frac{1}{2}$ arc CB. Angle C_1 is the inscribed angle measured by $\frac{1}{2}$(arc BZ + arc ZA). Since BZ = CB, angle C_1 = angle B_1 + angle A_2; also, angle E_1 = angle B_1 + angle A_2. But angle E_1 = angle B_2 + angle A_2, because E_1 is an exterior angle. Hence, angle B_1 = angle B_2. Now triangle BEA $\cong$ triangle BZA by s.a.s., also, AE = AZ; hence, AE = DC, and AE + EH = CD + CH. Therefore, AH = DC + CH.

To show the relationship between this theorem and the trigonometric formulas for the sum and difference of angles, it will be necessary to alter the figure somewhat. Consider Fig. 4(b).

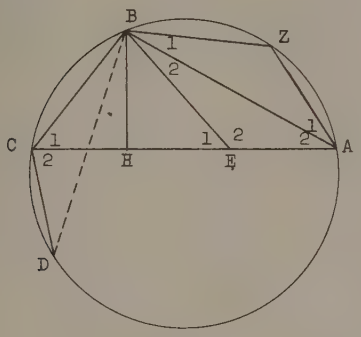

Fig. 4(a)

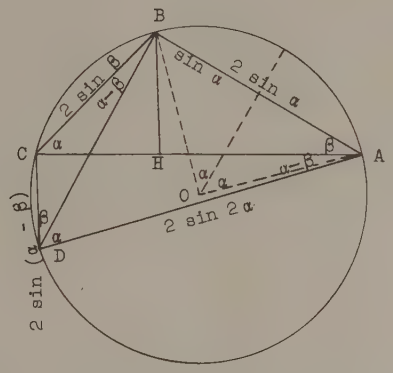

Fig. 4(b)

Angle DBA = $180^\circ - 2\alpha$, so sin DBA = sin 2α.

CA = 2 sin $(\alpha + \beta)$.

In triangle CBH, BH = 2 sin α sin β.
$\overline{CH}^2 = 4 \sin^2 \beta - 4 \sin^2 \alpha \sin^2 \beta$,
$\overline{CH}^2 = 4 \sin^2 \beta \cos^2 \alpha$,
CH = 2 sin β cos α.

Similarly in triangle ABH, AH = 2 sin α cos β

According to the theorem, CD = BC - AH; after substituting the trigonometric values into this equation the result is: sin $(\alpha - \beta)$ = sin α cos β - sin β cos α.

It is also true that AC = CH + HA; after making a similar substitution the result is: sin $(\alpha + \beta)$ = sin α cos β + sin β cos α.

The corresponding formulas for the cosine can be determined in a similar way by making angle ADB equal to $90^\circ - \alpha$.

No mathematical writings of HIPPARCHUS OF NICAEA (fl. 150 B.C.) are extant; but Theon of Alexandria (ca. 365 A.D.) claims that Hipparchus was the originator of trigonometry, and that he computed a table of chords.[18] This would almost imply that he was familiar with either the so-called Ptolemy's Theorem[19] concerning an inscribed quadrilateral, or as Carl Schoy[20] has shown, the theorem of the "broken chord" of Archimedes. Hipparchus was probably the first Greek astronomer to use degrees.[21]

18. Florian Cajori, op. cit., p. 47.
19. "The product of the diagonals of an inscribed quadrilateral equals the sum of the products of the opposite sides."
20. Op. cit., p. 3.
21. George Sarton, op. cit., p. 193.

Because of his extreme conservatism and timidity, Hipparchus was prevented from rejecting the geocentric system; for its long predominance, he is especially responsible.[22]

THEODOSIUS (ca. 60 B.C.) was the author of Sphaerica, a very complete treatise in three books on the geometry of the sphere. It contains no trigonometry; a spherical triangle is not mentioned.

MENELAUS OF ALEXANDRIA (ca. 98 A.D.) wrote six books on calculations of chords and three on spherics; the first is not extant, the second has been handed down in Arabic and Hebrew translations. Menelaus was the first to disengage trigonometry from stereometry and astronomy.[23] The first proposition in his third book on spherics is the so-called "Theorem of Menelaus"[24] relative to both spherical and plane triangles - this is sometimes called the regula sex quantitatum: "The product of the three ratios of the consecutive segments of the sides of a plane triangle made by any rectilinear transversal equals unity."

Let the transversal intersect the sides of triangle ABC in the points D, E, and F. Dropping from A, B, C perpendiculars AM, BN, CP to line FD, we have by similar triangles AD/DB = AM/BN, BE/EC = BN/CP, CF/FA = CP/AM. Whence by multiplication AD/DB x BE/EC x CF/FA = 1,

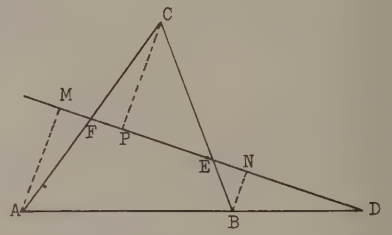

Fig. 5

It is to be observed that the figure is a complete quadrilateral. When applied to a spherical quadrilateral the theorem becomes: "The ratio of the chord (2 AD) to chord (2 DB) is composed of the ratio of chord (2 CE) to chord (2 BE) and of the ratio of chord (2 FA) to chord (2 CF)."

The Almagest of CLAUDIUS PTOLEMAUS (fl. 140 A.D.) controlled the science of astronomy for more than a millenium. It contained a splendid exposition of trigonometry, spherical and plane. It introduced the use and appreciation of approximate calculations.

The following selection is from Book I:[25]

On the Size of Chords Inscribed in a Circle

With an eye to immediate use we shall now make a tabular exposition of the size of these chords by cutting the circumference into 360 parts, and setting out the chords, with the arcs subtended by them increasing by half part. That is the diameter of the circle will be cut into 120 parts for the sake of ease in calculation, and we shall take the arcs, considering them with respect to the number they contain of those parts of which the circumference has 360, and compare them with the chords subtending them by finding out the number the chords contain of those parts of which the diameter has 120.

Ptolemy employed the sexagesimal system of the Babylonians. He assumed the ratio of the circumference of a circle to its diameter to be $3^{\circ}\ 8'\ 30''$, i.e., $\pi = 3 + 8/60 + 30/60^2 = 377/120 = 3.14166$.[26]

In his geometrical theorems equivalent to the addition formulas for the sine and cosine and in his computation of a table of chords, Ptolemy all but made a distinct

22. Idem.
23. Bjornbo, "Studien über Menelaos' Sphärik," Abhandlungen zur Geschichte der Mathematischen Wissenschaften, 14:10, Leipzig, 1902.
24. Idem.
25. Translation of the Almagest by Taliaferro (Annapolis, Maryland: St. John University Press, 1936), part 1, p. 18.
26. John David Bond, op. cit., p. 301.

mathematical science of trigonometry; this failure of trigonometry to attain its freedom was due to Ptolemy's lack of algebra and the cumbersome computational methods of that time.[27]

4. HINDUS

The greatest scientific advance of the first half of the fifth century was the introduction of the notion of sine.[28] Concerning this function Professor Karpinski states:

> By a stroke of genius the Hindu considered the ratio of the half chord to the radius as fundamental. Our whole modern trigonometry rests upon this foundation. The tables of the sine function have been computed from the chords in Ptolemy's Almagest. However, the new conception was absolutely Hindu.[29]

It is also a significant fact that the earliest computation with shadows upon which the tangent function depends, are found in India.[30]

The earliest works of the Hindus dealing with astronomy are the five theoretical treatises called Siddhāntas.[31] The mathematical section of the Paulisa Siddhānta contains the essence of Indian trigonometry. Kaye[32] has expressed the rules of this Siddhānta in modern notation:

$$\pi = \sqrt{10}$$

$$\text{Sin } 30^\circ = \tfrac{1}{2}, \quad \text{Sin } 60^\circ = \sqrt{1 - \tfrac{1}{4}}$$

$$\text{Sin}^2 \gamma = [(\text{Sin } 2\gamma)/2]^2 + [\{1 - \text{Sin } (90^\circ - 2\gamma)\}/2]^2$$

Here the radius is divided into 120 parts as opposed to the 60 parts of Ptolemy; this division enabled the table of chords to be converted into sines without numerical change.[33]

ĀRYABHAṬA (b. 476) wrote a treatise which is essentially a systematization of the results contained in the Siddhāntas.[34] He gave another measure for the radius (3438'), thereby enabling the sines to be expressed in a sort of circular measure.[35]

5. EASTERN ARABS

In the early part of the ninth century the caliph of Bagdad gathered together men of learning and set them to translating into Arabic the scientific works of the Hindus and Greeks; before the close of the century these treasures of learning were not only made accessible to the Arabs, but they also served as an impetus for original research.[36]

AL-KHOWARIZMI (d. ca 850) was one of the greatest scientists of his race and the greatest of his time.[37] "He influenced mathematical thought to a greater extent than any other medieval writer."[38]

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27. E. T. Bell, The Development of Mathematics (New York: McGraw-Hill, 1940), p. 55.
28. George Sarton, op. cit., p. 387.
29. L. C. Karpinski, "The Unity of Hindu Contributions to Mathematical Science," Scientia, :384, June, 1928.
30. Idem.
31. George Sarton, loc. cit.
32. Indian Mathematics (Calcutta: Thacker, 1915), p. 10.
33. Ibid., p. 11
34. George Sarton, op. cit., p. 409.
35. G. A. Kaye, loc. cit.,
36. John David Bond, op. cit., p. 307.
37. George Sarton, op. cit., p. 563.
38. Idem.

> ...The trigonometric tables in Latin translations by Adelard of Bath appear to be the earliest of Al-Khowarizmi's works to be known in Europe, indeed one of the earliest mathematical treatises taken from the Arabs; it was translated in 1126 A.D.[39]

Although it has been pointed out by Suter[40] that this translation is not of the original work by Al-Khowarizmi, but is based on Maslama-al-Majrītī's (ca. 1000 A.D.) revision of that work, still it is of considerable importance for the history of trigonometry, even though it is thought probable that the tangent function which appears in it was added by Maslama.[41]

Although AL-BATTĀNĪ (fl. 929) followed the older Arabs in combining Greek and Hindu developments, in his Opus astronomicum he used "sinus," "sinus complementi" or "cosinus," and "sinus versus."[42]

> The third chapter of his astronomy is devoted to trigonometry. He used sines regularly with a clear consciousness of their superiority over the Greek chords. He completed the introduction of the functions 'umbra extensa' and 'umbra versa'[43] (whence our cotangents and tangents) and gave a table of cotangents by degrees. He knew the relation between the sides and angles of a spherical triangle which we express by the formula cos a = cos b cos c + sin b sin c cos A.[44]

ABŪ-L WAFĀ' (940-998) invented a method for computing tables of sines which gave the sine of half a degree correct to nine decimal places.[45] "He was probably the first to show the generality of the sine theorem relative to spherical triangles."[46] He made a special study of the tangent and introduced the secant and cosecant; he knew the relation between the six trigonometric lines.[47]

Carra de Vaux, who made a study of the Almagest of Abūl-l Wafā', says that this treatise is divided into three parts, the first of which contains trigonometry; the second, the application of the formulas of trigonometry to observations; and the third, the theory of planets.[48]

In his definition of tangent and cotangent, Abūl Wafā' considers the radius equal to unity.[49]

AL-KHUJANDĪ (d. ca. 1000) and ABŪ NASR (ca. 1007) contend with Abū-l Wafā' for the priority in generalizing the Sine Law; the former was primarily an astronomer, the latter was the teacher of Al-Birūnī.[50]

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39. L. C. Karpinski, Robert of Chester's Latin Translation of the Algebra of Al-Khowarizmi (New York: Macmillan Co., 1915), pp. 17-18.
40. "Die Mathematiker und Astronomen der Araber und ihre Werke," Abhandlung zur Geschichte der Mathematischen Wissenschaften, 10:20, Leipzig, 1900.
41. L. C. Karpinski, loc. cit.
42. John David Bond, loc. cit.
43. Measurements by "shadows" were employed by the Hindus who computed the horizontal shadow of a stick of length 12 placed in a vertical position; this really gives the cotangent function. The Arabs extended this work by using a "second shadow," the reverse shadow, obtained by placing a stick of length 12 or some other length horizontally in a vertical wall. The "first shadow" was also indicated as the "regular shadow."
44. George Sarton, op. cit., p. 603.
45. Florian Cajori, op. cit., p. 105.
46. George Sarton, op. cit., p. 666.
47. Ibid., p. 667.
48. M. Le Baron Carra de Vaux, "L'Almageste d'Abu'l Wefa Albûzdjani," Journal Asiatique, 3rd series, 19:415, May-June, 1892.
49. Ibid., p. 420.
50. George Sarton, op. cit., pp. 667 and 702.

AL-BĪRŪNĪ (973-1048), a Persian, wrote a trigonometric treatise with the title, The Key to the Knowledge of Spherical Figures and Other Figures, which was used extensively by Naṣir ad-Dīn in his Treatise on the Quadrilateral.[51] In fact, the first appearance of the explicit statement of the Sine Law with its accompanying proof which found its way into so many trigonometric texts of the sixteenth century has for a long time been attributed to Naṣir ad-Dīn. Recent work in Arabic scientific literature reveals that the credit for being the first to state and prove the Sine Law for plane triangles should be ascribed to Al-Bīrūnī. Tropfke[52] and Schoy[53] both give the proof as Al-Bīrūnī gave it; the translation of Schoy's proof is given here: "The sides of a triangle have the same ratio as the sines of the opposite angles."

Prolong the sides of the triangle and with A as center construct an arc with the radius equal to unity. The sine falls between AB and AG of the chord AG. The arc GT is known because it measures angle BAC, and GM is the sine of this angle. Now with C as center and the radius equal to unity describe the arc RJ; then RK is the sine of angle BCA. Further

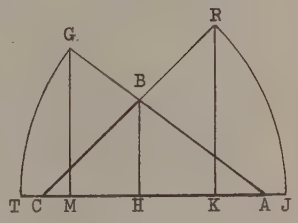

Fig. 6

$BH \perp AC$, and hence

$\triangle ABH \sim \triangle AGH$ so

$$AB:BH = AG:GM \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$\triangle CBH \sim \triangle CRK$ so

$$BH:BC = RK:CR \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

But $CR = AG$

The result of multiplying the ratios (1) and (2) is

$$\frac{AB}{BC} = \frac{RK}{GM}.$$

The Treatise on the Quadrilateral by NAṢIR AD-DĪN AṬ ṬŪSĪ (1201-1274) is the first known work in which trigonometry is treated independently of astronomy. In this treatise the ancient theory of the quadrilateral reached its highest development.

Inasmuch as it is quite evident that the work of Naṣir ad-Dīn influenced Regiomontanus and other European writers of the Renaissance Period, it seems imperative that a detailed account of this Arabic work should be given here.

Naṣir ad-Dīn, following the Arabic custom, prefaces his work with an invocation to God. He begins with the words, "In the name of the clement and merciful God," and proceeds to ask for blessings on the memory of His Prophet and the pious and virtuous members of His family. He then sets forth the purpose of his work - to translate into Arabic a treatise which he had written in Persian some years previously.

The work is divided into five books dealing with compound ratios and plane and spherical quadrilaterals. Chapters of varying length and number form the subdivisions of each book. Approximately two hundred sixty-five figures illustrate the text; summaries in tabulated form occur at intervals.

It is difficult to say whether the peculiar terminology in the treatise is due to the author or translator. However it is well to recall that Naṣir ad-Dīn based his Treatise on the Quadrilateral on the theorem of Menelaus; hence, a better understanding of the Arabic work is obtained, if with each definition a corresponding statement in the more familiar terminology of that theorem is given.

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51. Traité du Quadrilatère attribué à Nasiruddin-el-Tousey d'après un Manuscript tiré de la Bibliotheque de S. A. Edhem Pacha (Constantinople: Osamnie, 1881).
52. Geschichte der Elementar-Mathematik (2nd ed. Berlin: De Gruyter and Co., 1923), vol. V, p. 73.
53. Op. cit., pp. 53-54.

Lines which are not parallel or superposed are columns of a figure; AB, AC, CE, and BD are columns of Fig. 7. Each column comprises three lines determined by three points.

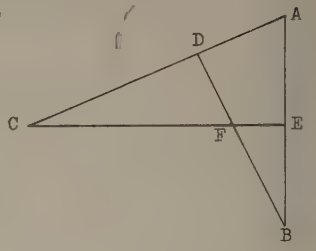

Fig. 7

By way of explanation, consider Fig. 7. ABCDEF is a complete quadrilateral involving four distinct triangles; each triangle has one transversal. The "column" of Naṣir ad-Dīn is in reality a transversal. AB is the transversal for the triangle CDF; AC, for triangle BEF; CE, for triangle ABD; and BD, for triangle ACE.

Two lines are called associated when they enter into a ratio, the one as consequent, the other as antecedent.

Each of the twelve lines in Fig. 7 is the associate of five lines. The association between two lines has place in three cases:

(1) When two lines are a prolonging of one by the other.
(2) When they form the sides of an angle of a triangle.
(3) When they fall between two columns.

For two lines to fall between two columns, it is necessary that they be the opposite sides of a quadrilateral. In Fig. 7, which is a complete quadrilateral, there are three quadrilaterals, namely: ADEF with opposite sides AE, DF and DA, FE; DBEC with the opposite sides EB, CD and CE, DB; and CABF with CA opposite FB and CF opposite AB.

There exist two associated lines according to (1), two according to (2), and 1 according to (3).

For example, AE in Fig. 7 is an association of the first kind with the lines AB and EB; an association of the second kind with the lines AC and EF; and an association of the third kind with the line DF. Therefore, each of the twelve lines in Fig. 7 is the association of five lines.

By observing Fig. 7, these lines are readily found, but in order to show that they form a ratio with AE, it is necessary to consider the four triangles of Fig. 7 with their transversals:

	TRI-ANGLE	TRANS-VERSAL	APPLICATION OF THEOREM OF MENELAUS	NAṢIR AD-DĪN'S EXPRESSION FOR THIS THEOREM
(a)	CDF	AB	CA/AD.DB/BF.FC/CE = 1	CA/AD = BF/DB x CE/FC
(b)	BEF	AC	BA/AE.EC/CF.FD/DB = 1	BA/AE = CF/EC x DB/FD
(c)	ABD	CE	BE/EA.AC/CD.DF/FB = 1	BE/EA = CD/AC x FB/DF
(d)	ACE	BD	AB/BE.EF/FC.CD/DA = 1	AB/BE = FC/EF x DA/CD

To verify the ratio for the association of the first kind, consult (b) and (c) in the foregoing table: AE is the consequent of BA and BE respectively.

For the association of the second kind observe that (c) can also be written as: AC/EA = CD/BE x FB/DF, whereby EA is the consequent of AC. If (b) and (c) are multiplied and the terms rearranged, the following is true: EF/EA = CD/DA x FC/AB x FB/DF and EA is the consequent of EF.

To obtain the ratio for the association of the third kind, express (c) as: DF/EA = CD/AC x FB/BE.

Those lines which do not give place to a ratio whereby one is the consequent and the other the antecedent, are called dissociate lines. In Fig. 5 each line is the dissociate of six others.

Consider the column AB in Fig. 8, supposing a proportion is set up, the first term of which is BE/EA. This ratio is an explicit ratio because:

(1) Neither of these two lines is a prolongation of the other, i.e., the ratio is not superposed.
(2) The point E, common to the two terms is not an extreme end of the column.
(3) B is the point proper to the antecedent and A is the point proper to the consequent.

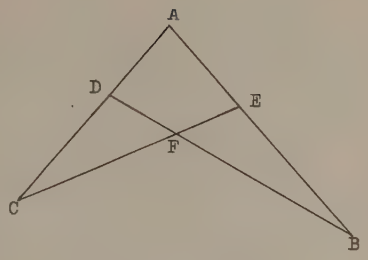

Fig. 8

Now consider BA/AE; this ratio is an implicit ratio for:

(1) The ratio is superposed.
(2) The common point is an extreme end of the column.
(3) B is the point proper to the antecedent and E is the point proper to the consequent.

Considering in Fig. 8, the column AB, and the explicit ratio BE/EA, it is seen that CE is the column that intersects AB at the point common to the two terms of the explicit ratio; for this reason CE is called the inactive column.

The inactive column contains the points C, E, and F; the other three vertices of the quadrilateral, namely A, B, and D, determine the inactive triangle.

The six lines AB, BD, AD, CE, EF, and FC are the inactive lines. The remaining six lines make up the double ratio: BE/EA = BF/FD x DC/CA.

By comparing with (c) in the table on page 18, it is perceived that Naṣir ad-Dīn is here applying the theorem of Menelaus to the triangle ABD, whose transversal, which he calls the inactive column, is CE.

Naṣir ad-Dīn quotes Arabic as well as Greek authors. Unlike so many medieval writers who fail to state their sources, he takes it upon himself "to give credit where credit is due." It is not necessary at this time to enumerate all references to Greek authors; a few instances have been chosen merely to demonstrate Greek influence on Arabic works.

Five references are made to the Elements of Euclid. Naṣir ad-Dīn places as an "advertisement" to the propositions of Book I, Euclid's definition of composition and decomposition of ratios:

> We read at the beginning of Book VI of the Elements of Euclid that a ratio is composed of other ratios when the quotients of these ratios being repeated, the one by the other, a certain ratio results; there are some ratios which are decomposed into other ratios, when their ratios are dividing the one by the other in order to form certain other ratios.

Abu Nasr, the author of the Royal Almagest, wrote a commentary on the Spherics of Menelaus; it is to this and not to the Greek work itself, that Naṣir ad-Dīn makes his references.

Naṣir ad-Dīn quotes Ptolemy frequently and at great length. This is not surprising, for the Almagest played no small part in Arabic science. This reference from Book II deals with implicit ratios in a spherical quadrilateral:

> From the implicit ratio and its demonstration, Ptolemy was content to prolong two columns of the quadrilateral which he proposed, in order to complete two half-circumferences. In this manner he formed a new quadrilateral by means of which he demonstrated the explicit ratio. He proved the following proposition by means of the explicit ratio.

Thus, let ACFB be the proposed quadrilateral; the implicit ratio is

$$\frac{\sin BA}{\sin AE} = \frac{\sin BD}{\sin DF} \times \frac{\sin FC}{\sin CE}.$$

Demonstration:

Prolong both of the columns BA, BD so that they intersect in a manner to complete the semicircumferences BAH and BDH. The explicit ratio of the quadrilateral CEHD is sin HA/sin AE = sin HD/sin DF x sin FC/sin CE; but sin HA = sin BA and sin HD = sin BD, whence sin BA/sin AE = sin BD/sin DF x sin FC/sin CE.

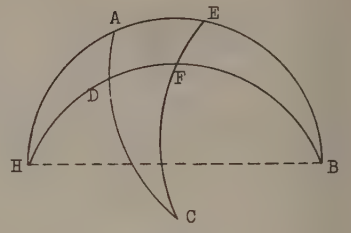

Fig. 9

This theorem also implicitly shows the application of the theorem of Menelaus to a spherical triangle. Of course the sine function was not known to Menelaus; the expression that he would employ for the last proportion is: chord (2 BA)/chord (2 AE) = chord (2 BD)/chord (2 DF) x chord (2 FC)/chord (2 CE).

Naṣir ad Dīn gives two methods of proof of the Sine Law for both the acute and obtuse triangle. His second proof[54] is:

Obtuse

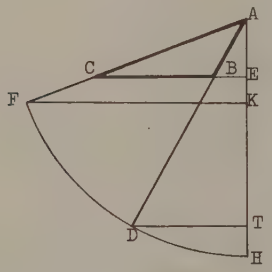

Fig. 10(a)

Acute

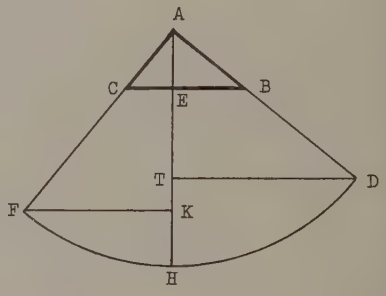

Fig. 10(b)

Let ABC in the above figure be the given triangle. Make AE perpendicular to BC. Prolong AB and AC so that AF = AD = 60 = R. Describe the arc DH. Make FK and TD perpendicular to AH. In the triangle ABE, the angle E being a right angle, B will be the complement of A; DT = sin A; AT = sin B. In the triangle AEC, FK = sin A; KA = sinc C. Because of the similarity of the two triangles ABE and ADT,

$$\frac{AB}{AE} = \frac{AD \text{ or radius}}{AT \text{ or sin } B} \quad \text{and} \quad \frac{AE}{AC} = \frac{AK \text{ or sin } C}{AF \text{ or radius}} .$$

Because of the similarity of the two triangles AEC and AKF, it is possible to deduce the statement:

$$\frac{AB}{AC} = \frac{AK \text{ or sin } C}{AT \text{ or sin } B} .$$

The Treatise on the Quadrilateral also contains the six fundamental formulas for the solution of spherical right-angled triangles. It shows how to solve other triangles as well, replacing, if necessary, the consideration of the angles by that of sides and vice versa, by means of polar triangles. Nasir ad-Dīn does not define polar triangles, but his method implies their consideration. Book V offers this demonstration: "Solution of a spherical triangle when three angles are known." An interpretation of Naṣir ad-Dīn's proof will also be given.

54. The first proof is the one that originated with Al-Bīrūnī; see p. 9.

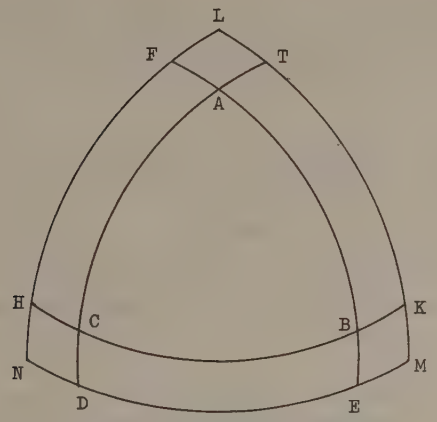

Fig. 11

In the triangle ABC prolong AB and AC so that AD and AE become quadrants. In a similar manner construct the quadrants BF, BH, CT, and CK. Draw the arcs of great circles DE, TK, and FH, the points of intersection of which will be L, M, and N. The angles A, B, and C being known, DE, TK, and FH are known. K and H being two right angles, L will be the pole of KH; similarly M will be the pole of TD, and N that of FE. Now TL and KM will be known because they are complements of KT and LK respectively. LN and MN will also be known. Thus the three sides of the triangle LMN will be determined. With this information now at hand apply the previous case, namely: 'Solution of a spherical triangle when three sides are given.' Thus the arcs KH, DT, EF will be known. But each of the arcs KC and BH being equal to a quadrant, the complement of HK will be equal to BC, hence BC will be known. In a similar manner AB and AC can also be determined. By this process we arrive at the knowledge of the sides of the triangle ABC.[55]

With the vertices of the spherical triangle ABC as poles construct three great circles. The great circles whose poles are B and C will intersect in two diametrically opposite points. Denote by L the point of intersection which is on the same side of BC as is A. Determine M and N similarly. Then LMN is the polar triangle of ABC; conversely, ABC is the polar triangle of LMN. Each angle of a spherical triangle is the supplement of the corresponding side in the polar triangle. Hence:

$$C + LM = 180^{\circ}.$$

But C is known, so LM is known. LN and MN will be known for a similar reason.

From a possible forty or fifty Arabic writers of trigonometry, the following ones might be mentioned. ḤABASH AL ḤĀSIB (the calculator), who flourished in Bagdad about 850, compiled three astronomical tables and seems to have introduced the notion of the "shadow" equivalent to the tangent; he compiled a table of such "shadows," which seem to be the earliest of its kind.[56]

AL-NAIRĪZĪ, who was active from 892-902, also used the "shadow" function as a genuine trigonometric line.[57]

55. See pp. 28-29 for some other theorems of Naṣir ad-Dīn which are compared with similar ones of Regiomontanus.
56. George Sarton, op. cit., p. 565.
57. Ibid., p. 599.

IBN YŪNUS (d. 1009) in Cairo, was the author of the Hakemite Tables which were the sum total of two hundred years of Arabic astronomy and were used as an unfailing authority of all later astronomers of the Orient.[58] He was the first to introduce those prosthapheretical formulas which were indispensable before the invention of logarithms, namely, the equivalent of

$$\cos a \cos b = \tfrac{1}{2}[\cos (a - b) + \cos (a + b)].$$[59]

ABŪ-L-HASAN KŪSHYĀR IBN LABBĀN, prominent from about 971 to 1029, continued the investigations of Abū-l-Wafā' on the tangent.[60]

The last and probably the greatest of the Arabic astronomers was ULUG BEG (1393-1449); he found sin 1° by means of his famous third-degree equation, $x = \frac{r^2/4 \sin 3^\circ + x^3}{3\, r^2/4}$, in which he replaced x by sin 1° and solved by a method of approximation.[61]

6. WESTERN ARABS

Political animosities severed the intercourse between the Eastern Arabs and the Western Arabs in Spain. Science in the East was far in advance of that in the West. Since the Arabs in Spain sponsored the progress of mathematics in medieval Europe, the works of Abū-l Wafā' and Naṣir ad-Dīn failed to gain recognition for nearly three centuries.

AL-ZARQĀLĪ (ca. 1061-1080), one of the most celebrated astronomers of his time, discovered many errors in the Almagest; as a consequence he subjected the Ptolemaic theory to severe criticism.[62] He assembled a group of Jewish and Moorish scholars who, working under his direction, constructed the Toledo Tables. Of special interest is the fact that he defined the "sinus totus" as the semi-diameter of 150.[63]

JĀBIR IBN AFAH (fl. ca. 1145), also known as Geber, wrote an astronomical treatise, Islah al-Majesti, of nine books. The trigonometrical introduction was of special importance, for it contained a new method of solving right-angles spherical triangles.[64] The formula for this solution, often called "Geber's Theorem," is: "If a, b, c are the sides and A, B, C, the angles of a spherical triangle, right angle at A, then cos B = cos b sin C."[65]

The Alphonsine Tables[66] were constructed about this time. They contained a sine table similar to that of Al-Battānī, also a table of tangents for the gnomon equal to twelve "fingers."[67]

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58. A. von Braunmühl, op. cit., p. 62.
59. George Sarton, op. cit., p. 717.
60. Idem., vol. I., p. 717.
61. John David Bond, op. cit., p. 304.
62. George Sarton, op. cit., p. 123.
63. See Maximilian Curtze, Aus den Canones sive regule super tabulas Toletanas des Al-Zarkâlî in "Urkunden zur Geschichte der Trigonometrie im Christlichen Mittelalter," Bibliotheca Mathematica, 1:337-347, Leipzig, 1900.
    See also, Ernst Zinner, "Die Tafeln von Toledo (Tabulae Toletanae," Osiris, 1:745-774. January, 1936.
64. George Sarton, op. cit., p. 709.
65. Florian Cajori, op. cit., p. 109.
66. These tables were founded on the theory of Ptolemy and the observations of the Arabs; they were computed by a number of Arabian and Jewish astronomers under the personal direction of King Alphonse, and were printed at Venice in 1483 (Braunmühl, op. cit., p. 100).
67. Idem.

7. EUROPEANS

The outline of trigonometry in the twelfth and thirteenth centuries gives a high idea of Moslem science. Latin trigonometry was scarcely a reflection of the Arabic; it was always a little behind the times, even when it was new, for Arabic efforts did not cease, but continued with augmented efficiency.[68]

Professor Karpinski frequently emphasizes the fact that, "The transmission to Europe of the science of the Orient and the near East was effected over several centuries by Latin and Hebrew translations."[69] More than this, some of the material was also translated into French.[70]

The Englishman, ADELARD OF BATH (fl. ca. 1125), translated the astronomical tables of Al-Khowarizmi, as revised by Maslam,[71] which introduced the sine and tangent functions into the West.[72] Another translation was probably made by HERMAN THE DALMATIAN; tables for London were compiled by ROBERT OF CHESTER, who also revised Adelard's translation.[73]

PLATO OF TIVOLI (fl. ca. 1140) deserves special credit for his translation of Al-Battani's De motu stellarum.[74] "Out of this translation sprang the word 'sinus' as the name of a trigonometric function."[75]

Hearing that in Spain he would have access to Ptolemy's Almagest, GERARD OF CREMONA (1114-1187) left his native Italy and joined the school of translators at Toledo. Besides the Almagest, he also made translations of the works of Menelaus, Al-Zarqālī, and Geber.[76]

To England belongs the honor of having produced the earliest European writers on trigonometry; the works of BRADWARDINE, RICHARD OF WALLINGFORD,[77] and SIMON BREDON OF WINCHECOMBE contain trigonometry gathered from Arabic sources.[78]

The Practica Geometriae of LEONARD OF PISA, also called Fibonacci, published in 1220, is the first treatise in which surveying by trigonometric methods is placed on a mathematical basis.[79]

Graeco-Arabic trigonometry was continued (probably independently of Naṣir ad-Dīn) in Hebrew by LEVI BEN GHERSON[80] (ca. 1321), also known as Leo de Balneolis. In his De

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68. George Sarton, op. cit., p. 12.
69. L. C. Karpinski, "Descartes and the Modern World," Science, 89:151, February, 1939.
70. See for instance: L. C. Karpinski, "Two Twelfth Century Algorisms," Isis, 3:396-413, 1921; also, "An Anglo-Norman Algorism of the Fourteenth Century, Isis, 23:121-151, June, 1935.
71. See p. 8.
72. George Sarton, op. cit., p. 125.
73. Ibid., p. 126.
74. Plato's translation was published in Nuremberg, 1537, together with Al-Farghani's Elements: Albategenius de motu stellarum ex observationibus tum propriis tum Ptolemaei, omnia cum demonstrationibus geometricis et additionibus Ioannis de Regiomonte, Nuremberg, 1537 (Ibid., p. 602). See also C. A. Nallino, Publicazione del Reale osservatorio di Brera, XL, 1-3, 1899-1903.
75. Florian Cajori, op. cit., p. 104.
76. John David Bond, op. cit., p. 315.
77. John David Bond, "Richard Wallingford's Quadripartitum" (English Translation), Isis, 5:340-363, October, 1923.
78. Florian Cajori, op. cit., p. 127.
79. Boncompani, Scritti di Leonardo Pisano del Secole Decimoterzo (Rome, 1862); Practica Geometriae is in the second volume of the Scritti.
80. See for instance: Joseph Carlebach, Levi ben Gerson als Mathematiker (Berlin: L. Lamm, 1910).

sinibus chordis et arcubus,[81] the Sine Law appears for the first time in the mathematical literature of Europe.[82]

JOHANNES DE LINERIIS (fl. 1322) learned astronomy with great success at Paris;[83] he wrote a spherical trigonometry.[84] JOHANNES DE MURIS compiled a Table of Sines wherein he imitated Al-Zarqālī by using a semi-diameter of 150. He shows some independence, however, by constructing a figure of his own in place of the traditional one of Al-Zarqālī.[85]

The radius of 600 000 originated with JOHN SCHINDEL, also called John of Gmunden, (1380?-1442).[86] This invention was adopted and furthered by his immediate successors, the foremost among them being PEURBACH and REGIOMONTANUS. Concerning the latter, von Braunmühl has aptly said:[87]

> In the short space of time that was granted this man for the operation of his excellent capabilities, he accomplished so much that he surpassed all his contemporaries and stands as a monument at the beginning of a new era.

81. Part of the Latin text is given by Maximilian Curtze, "Urkunden zur Geschichte der Trigonometrie in christlichen Mittelalter," Bibliotheca Mathematica, 1:372-380, Leipzig, 1900.
82. John David Bond, "The Development of Trigonometric Methods down to the Close of the XVth Century," Isis, 3:296, October, 1921.
83. A. von Braunmühl, op. cit., p. 107.
84. Maximilian Curtze, "Canones tabularum primi mobilis," op. cit., pp. 391-413.
85. Maximilian Curtze, "Sinusrechnung des Johannes de Muris," ibid., pp. 413-416.
86. John David Bond, loc. cit.
87. A. von Braunmühl, op. cit., p. 119.

Tabule directionū ꝓfectionūq; famosissimi viri Magistri Joannis Germani de Regiomonte in nativitatibus multum vtiles.

Tabula Secunda

Numerus		Numerus		Numerus	
0	00000	31	60086	61	180402
1	1745	32	62486	62	188075
2	3492	33	64940	63	196263
3	5240	34	67452	64	205034
4	6992	35	70022	65	214450
5	8748	36	72654	66	224607
6	10511	37	75356	67	235583
7	12278	38	78129	68	247513
8	14053	39	80978	69	260511
9	15838	40	83909	70	274753
10	17633	41	86929	71	290422
11	19439	42	90040	72	307767
12	21256	43	93254	73	327088
13	23087	44	96571	74	348748
14	24932	45	100000	75	373211
15	26794	46	103551	76	401089
16	28674	47	107236	77	433148
17	30573	48	111062	78	470453
18	32492	49	115037	79	514438
19	34433	50	119177	80	567118
20	36396	51	123491	81	631377
21	38387	52	127994	82	711569
22	40402	53	132704	83	814456
23	42448	54	137639	84	951387
24	44522	55	142813	85	1143131
25	46631	56	148253	86	1430103
26	48772	57	153987	87	1908217
27	50952	58	160035	88	2863563
28	53170	59	166429	89	5729796
29	55432	60	173207	90	Infinitū
30	57734				

Plate 2. Regiomontanus Tabule directionum 1490, Augsburg, published by Ratdolt.

poli. 48. graduũ reperio. 8. gradus ꝛ. 21. minuta cancri. Horum duorũ arcuũ ecliptice dr̃a est. 55. minuta de quibus pars proportionalis scõm ꝓportionẽ 45. minutorum ad. 60. est. 41. minuta fere que adiecta arcui ecliptice ad. 47. gradus reperto conflant. 8. gradus ꝛ. 7. minuta cancri: arcũ scz ecliptice que situ ꝓ descensione demũ obliqua: ¶ Breue exemplũ accipe: offert mihi descẽsio obliqua 3. 97. graduũ ꝛ. 34. minutoꝝ cui correspondentẽ arcũ ecliptice iubeor inuestigare ad eleuationẽ poli. 48. graduum. Adiungo memorate descẽsioni. 180. gradus ꝛ resultant. 277. gradus ꝛ. 34. minuto hunc arcũ tãq̃ ascensionẽ obliquã offendo in tabula 48. graduum iuxta. 12. gradus sagittarij rectisq̃. 6. signis cõibus perducor ad. 12. gradus geminoꝝ qui videlz respondent oblate descensioni.

Decimũ probleuma.

Ascensionẽ obliquã stelle cuiuscũq̃ in orizonte quolibet diuinumerare: huic nostro ꝓposito seruiet tabula differentiarũ ascensionaliũ dum stella non pluribus q̃: 32. gradibus ab equatore remouet. In latere enim sinistro dicte tabule vtriusq̃ partis tam borealis q̃ australis declinatio vsq̃ ad. 32. gradus ponit q̃ multius planetarũ: quoruz gratia potissimũ tabulã ipsam: condidim⁹ tantã declinationẽ egredi soleat. In latere aũt superiori trãsuerso ponunt eleuationes poli septẽtrionalis supra orizontes regionuz ac circulos positionum vsq̃ ad. 60. gradus. Arca aũt tabule dr̃as ascensionũ cõplectit. Eleuationem igit poli quere in fronte tabule memorate: declinationẽ aũt stelle siue borealẽ siue australẽ in latere sinistro. Nam quod in angulo cõmuni offendes erit differentia ascensionũ stelle ꝓposite: quã demas ex ascensione recta stelle superius inuenta: si declinatio stelle borealis fuerit: aut eidem adicias si australis extiterit. Sic enim vel relinquet vel colliget ascensio obliqua stelle quam querebas ad orizontẽ ꝓpositũ. Q̃ si stelle fixe ampliorẽ q̃. 32. graduũ declinatiõ habentis ascensionis obliquã computare libeat. Intrabis tabellã secundã cũ eleuatione poli supra orizontem datũ ꝛ numerũ ex directo eius occurrentẽ seruabis. Similiter in eadẽ tabella accipies numerũ cum declinatione stelle ad quã cũq̃ partẽ fuerit hoꝝ numeroꝝ alter alterũ multiplicet ꝛ productũ per. 6. extendat. Reiectisq̃ primis. 6. figuris versus dextrã vnitatẽ relictis iungendo si reiecte plus. 500000. denotauerint relinquet sinus rectus differentie ascensionũ stelle ꝓposite. Cuius sinus arcũ docebit tabula sinuz maximũ habens 60000. particularũ. Cognita itaq̃ dr̃a ascensionũ veteris ad ascensionem obliquã stelle sciendam quẽadmodũ tam prĩdẽ monuimus. Facilius tamẽ idem efficies si tabula tua maximũ sinum habeat. 100000. Nam altero duorum numeroꝝ per eleuationẽ poli ac declinationẽ stelle inuentoꝝ in alterũ multiplicato a prodicto abijcies quinq̃ figuras primas versus dextram vnitatẽ relictis adiuncta si abiecte plus. 50000. repzesentauerint ꝛ reliqui sinus recti arcuz ex tabula memorate elicies qui erit differentia ascensionũ ꝓpositio tuo conducibilis. Dic tamen animaduertẽdũ q̃ quãdo differentia ascensionum ab ipsa ascensione recta subtrahi nequit adiciendus est integer circulus. 360. graduũ ꝛ ab aggregato minuenda est differentia ascensionũ prefata. Similiter quando differentia ascensionũ adiecta ascensioni recte numerũ maiorũ. 360. gradibus coaceruauerit ipsi. 360. gradus reiciendi sunt: relictis videlicet ꝓ ascensione obliqua cõputatis. ¶ In exemplo reperiẽ stella secundi probleumatis que habuit declinationẽ septentrionalem. 9. graduum ꝛ. 51. minutoꝝ volo inuenire ascensionem eius obliquaz in orizonte cui polus eleuat̃. 48. gradibus per tabulam igitur differentiarũ ascensionalium sub eleuatione poli. 48. graduuz cum declinatione. 9. graduũ ꝛ. 51. minutoꝝ duplici introitu inuenio differentiam ascensionũ. 11. graduum ꝛ. 7. minutoꝝ quam demo ex ascensione recta stelle inuenta per terciũ aut quartũ probleuma que. 164. gradus ꝛ. 34. minuta sic enim relinquit ascensio obliqua. 153. graduum ꝛ. 27. minutoruz. Scõm viam autem vniuersalem sic procedo. In tabella secunda iuxta numeruz eleuationis poli. 48. graduum inuenio numerum. 11062. quem seruo ad partẽ Item per eandem tabellam cuz declinatione stelle duplici introitu elicio aliũ numerum. 17364. quem duco in ꝓius seruatum numerum producunt. 1928480568. ille numerus per senarium multiplicatus reddit hunc. 11570883408 abiectis autem primis sex figuris adiectaq̃ vnitate vt assolet remanet mihi sinus differentie ascensionũ. 11571. cuius arcus est. 11. gradus ꝛ. 7. minuta cũ quo tandem vt ꝓius ascensio obliqua reperiat̃. Tabellam aũt sepe dictam non iniuria secundã appellare libuit q̃ multiuariã ac mirandẽ vtilitatẽ instar fecũ de arboris parere soleat.

Undecimũ probleuma

Ascensionẽ obliquam stelle cuiuscũq̃ numerare: inuenta differẽtia ascensionũ stelle ꝓposite ex precedenti documẽto addeeã ascensioni recte ipsius stelle declinationẽ septẽtrionalẽ habentis: vel minue huiusmodi differentiã ascensionũ ab ea si declinatio stelle fuerit meridiana. Nam quod colliget aut residuabitur erit descẽsio obliqua stelle ꝓposite. Hic nõ est opus exemplo aliquo speciali cum differentia ascensionũ que ꝓius addita est ꝓ ascensione obliqua habenda hic subtrahi ꝓ descensione obliqua ꝛ econtra que ibi subtrahi hic addi.

Duodecimũ probleuma.

Arcum semidiurnũ solis vel alterius stelle cuiuscũq̃ cognoscere inuenias modo predicto. 10. probleumatis differẽtiã ascẽsionũ stelle que est etiam dimidia dr̃a diei equinoctialis ꝛ diei inequalis stelle: eã igit. 90. gradib⁹ adiunge si stella declinationẽ septẽtrionalem habuerit: aut ex eis. 90. gradibus minue ꝓ stella declinationem habente meridianam: sic enim vel conflabis

Plate 3. Regiomontanus, 1490, Augsburg, published by Ratdolt.

Chapter II

THE REBIRTH OF TRIGONOMETRY - REGIOMONTANUS TO COPERNICUS

1. INTRODUCTORY NOTE - PROGRESS AND CULTURE

Human civilization or culture, being a gradual, complicated, and often irregular development,[1] is dependent on progress.[2] Human progress on the one hand, means an increase in knowledge; on the other a growing appreciation of the humanity of others.[3] Science and aesthetics are the respective expressions of this twofold direction of progress, whose advance and appreciation mark every great period of the world's history.[4] To this rule, the era which witnessed the rebirth of trigonometry sponsored by Regiomontanus, the greatest of the immediate predecessors of Copernicus, was no exception. The capture of Constantinople by the Turks in 1453, resulting in a dissemination of knowledge from the Eastern capital; the discovery of America in 1492, enlarging the sphere of human imagination and arousing the spirit of adventure; the popular use of great inventions, such as the mariner's compass, printing press, paper, and gunpowder; the revival of interest in the classical authors; and the deposition of the earth from its supposed central position in the Universe[5] - all these activities which impressing this period with a scientific and humanistic character, indicate that the progress of each science is dependent on the progress of the others.[6] Hence, it was no chance event that there was a rebirth of trigonometry; it was most natural that a Regiomontanus should bring about its revival.

2. REGIOMONTANUS - Biographical Sketch

Johann Müller, more generally known as Regiomontanus, was born in Königsberg, a small town in lower Franconia, June 6, 1436.[7] The name of his family agreed with the trade of the father who operated a mill.[8] At about the age of twelve, the young Johann entered the University of Leipzig to study dialectics; here he manifested the love for astronomy and mathematics that marked his studious career.[9]

At the age of fifteen, Regiomontanus went to Vienna.[10] He studied astronomy and trigonometry under Peurbach,[11] who showed him how incorrect were the Alphonsine Tables and how false were the Latin translations of the Greek astronomers from intermediate Arabic translations.[12] Peurbach set himself the task of composing an "Epitome" on

1. Lynn Thorndike, A Short History of Civilization (New York: F. S. Crofts and Co., 1933), p. 3.
2. C. J. Hayes, A Political and Social History of Modern Europe (New York: Macmillan Co., 1924), vol. I, p. 196.
3. F. S. Marvin, Progress and History (London: Oxford University Press, 1917), p. 7.
4. C. J. Hayes, loc. cit.
5. W. D. Wallis, Culture and Progress (New York: McGraw-Hill, 1930), p. 425.
6. George Sarton, op. cit., vol. I, p. 31.
7. J. H. Mädler, Geschichte der Himmelskunde (Braunschweig: Westerman, 1873), vol. I, p. 124.
8. F. Boquet, Histoire de l'Astronomie (Paris: Payot, 1925), p. 236.
9. Pierre Gassendi, Opera omnia (Florence: Apud Joannem Cajetanum Tartini et Sanctem Franchi, 1727), vol. V, p. 462.
10. F. Boquet, loc. cit.
11. George Peurbach was born in Austria in 1423; the cognomen is not his family name - his real surname is unknown. He took his Master of Arts degree "cum laude" at the renowned Academy of Vienna. He taught astronomy at Ferrara, and Padua; returning to Vienna, he became professor of mathematics and astronomy, succeeding his master, John of Gmunden (Pierre Gassendi, op. cit., p. 459).
12. Florian Cajori, op. cit., p. 131.

Ptolemy's Almagest, but he did not live to finish it.[13] Regiomontanus was a loyal co-laborer as long as his teacher lived; after Peurbach's death, he prepared the unfinished work of his master for publication.[14]

Regiomontanus learned the Greek language from Cardinal Bessarion,[15] who for a time resided in Vienna.[16] The Cardinal, returning to Italy, took the young scholar with him to Rome as a companion; here Regiomontanus found opportunity of becoming intimately acquainted with the Greek language, in which he afterwards became very proficient.[17]

During his stay in Italy, Regiomontanus continually made observations of the sun, the moon, and the planets and searched the libraries for Greek manuscripts.[18] Part of the time he traveled with Cardinal Bessarion, who as papal legate often changed his abode; occasionally he traveled alone through Italy visiting Viterbo, Ferrara, Padua, and Venice where he established scientific relations with the well-known astronomers, notably the ninety-year-old Bianchini.[19] It is noteworthy that in Padua in the year 1464 he lectured on the Arabic astronomer, Al-Farghānī, since he had in his possession a Latin translation of the astronomical work of this Arab.[20] In the same year he worked in Venice on De triangulis libri quinque which he had already begun in Rome and which later gave such a great impetus to the development of trigonometry.[21]

After two years absence from Rome, Regiomontanus returned there alone in October, 1464, to spend four more years in studying and copying.[22] He was able to point out grave errors in the work of George of Trebizond;[23] the consequent enmity of the latter and the absence of his patron seem to have induced Regiomontanus to leave Italy in 1468.[24]

Regiomontanus returned to Vienna to pursue private study. He accepted the invitation of Corvinus[25] to manage his rich library in Buda; here he produced his Tabulae

13. A. von Braunmühl, loc. cit.
14. Idem.
15. Cardinal Bessarion (1402-1472) was born and received his early education in Trebizond, the most important city of Asiatic Greece. His craving for learning caused him to desert his native city and to adopt Constantinople, Rome, and the leading cities of Italy as his home. He took Holy Orders, played an active part in the short-lived reunion of the Greek and Latin Churches - he left the Greek Church and joined the Latin rite. He did much to promote the Italian Renaissance, which is evidenced by the "Inventaire des manuscrits grecs et latins donnés à Saint Marc de Venise par le Cardinal Bessarion, 1468" in Revue des Bibliothèques, 4:129-187, 1894 (Henry Vast, Le Cardinal Bessarion Paris: [Hachette and Co., 1878]).
16. Florian Cajori, loc. cit.
17. A. von Braunmühl, loc. cit.
18. Idem.
19. In 1458 Giovanni Bianchini published the Novae tabulae coelestium motum which appeared after his death in four other editions (1495, 1526, 1553, 1575). At the request of Frederick III, he made a commentary of the Alphonsine Tables. See Maximilian Curtze, "Der Briefwechsel Regiomontan's mit Giovanni Bianchini, Jacob von Speier und Christian Roder," Abhandlungen zur Geschichte der Mathematischen Wissenschaften, 12: pp. 192-336, Leipzig, 1902.
20. A. von Braunmühl, loc. cit.
21. Idem.
22. Pierre Cassendi, op. cit., p. 466.
23. George of Trebizond, born on the Island of Crete 1396, came to Venice in 1430 as professor of Greek. He was the first to give a translation in Latin of the Greek text of the Almagest; he also translated some works of Theon (Boquet, op. cit., p. 233).
24. Pierre Gassendi, loc. cit.
25. Matthias Corvinus, King of Hungary, born in 1443 at Klausenburg in Transylvania, was the son of John Hunniade. His father died when he was thirteen. Political enemies killed his brother and took Matthias into Prague where he was detained until 1458, when the Hungarian nation claimed him for its king. Corvinus had a deep appreciation for learning. He profited by the dispersion of the Greek libraries after the fall of Constantinople to enrich his own library at Buda (J. F. Michaud, Biographie Universelle [Paris: 1854], vol. IX, pp. 281-282).

IOANNIS REGIO-

MONTANI MATHEMATICI PRAE-

STANTISSIMI DE TRIANGVLIS PLANIS ET SPHAERIcis libri quinque, unà cum tabulis sinuum, in quibus tota ip sorum triangulorum scientia ex primis fundamentis Geometricarum ἀποδείξεων absolutissimè extructa continetur.

QVAM MVLTIPLICEM VSVM HAEC TRIANGVlorum doctrina omnibus legitimè philosophantibus adferat non solùm ad expeditè absoluendas, quæcunque in locis terrestribus ac maritimis occurrãt, dimensiones, sed etiam ad intelligendos fontes eius disciplinæ, quæ extructa est à Ptolemæo & Copernico de Reuolutionibus orbium cælestium, qui sanarum intelligentia sunt instructi, in sequenti opere, quod complectitur ordinatã Astronomicorum & Geometricorum problematum descriptionem, tanquam in clara luce intueri & experientia infallibili duce deprehendere poterunt.

Omnia nunc primùm in lucem edita in gratiam Mathescos studiosorum per
DANIELEM SANTBECH Nouiomagum.

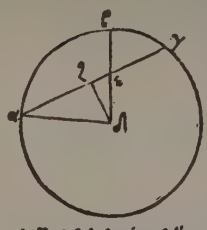
395

Cum Cæs. Maiestatis gratia & priuilegio.

1561.

BASILEAE.

δ δύτερον λύτλικον λαμμάτιον.



Περιφερειῶν δ' αὐτῶν ὄντων ἀνίσων, ἔστω μείζων ἡ α γ τῆς β γ περιφερείας, καὶ ὁ λόγος ὁ τῆς τοῦ ἡμικυκλίου τῆς α β πρὸς τὴν ἡμικυκλίου τῆς β γ, μείζονα τοῦ λόγου τῆς ἐκ τοῦ κέντρου τῆ α β καὶ β γ περιφερῶν. ἐπεζεύχθωσαν γὰρ αἱ τῆς περιφερείας, ἐπιζεύχθω ἡ α δ, καὶ ἐκβεβλήσθω ἐπὶ τὸ ε· καὶ πάλιν ἐπεζεύχθω ἡ α γ ἐπὶ τὸ ζ. ἔστι μὴν ἴση ἡ α γ περιφέρεια τῆ α δ ζ γωνίᾳ, τουτέστιν ἡμίσειαν αὐτῆ τετυχήκασι· διπλῆ γὰρ ἑκατέρα ἑκατέρας· καὶ ὅλη ἄρα ἡ α δ ζ πρὸς γωνίαν ὅλην τὴν α δ β ὡσαύτως ἔχει καὶ ἡ α γ περιφέρεια πρὸς τὴν α β περιφέρειαν· ἀναπέπονθε γὰρ ἑκατέρα ἑκατέρᾳ. ἀλλὰ μὴν καὶ ἡ μὲν α δ ζ γωνία πρὸς τὴν α δ β μείζονα λόγον ἔχει ἤπερ ἡ α γ περιφέρεια πρὸς τὴν α β περιφέρειαν· ἡ γὰρ α δ ζ γωνία πρὸς τὴν α δ β μείζονα λόγον ἔχει ἤπερ ἡ α ζ πρὸς τὴν α β· ἡ δὲ α ζ πρὸς τὴν α β μείζονα λόγον ἔχει ἤπερ ἡ α γ περιφέρεια πρὸς τὴν α β περιφέρειαν· πολλῷ δὲ μᾶλλον ἡ α δ ζ γωνία πρὸς τὴν α δ β μείζονα λόγον ἔχει ἤπερ ἡ α γ περιφέρεια πρὸς τὴν α β περιφέρειαν. ἔστι δὲ καὶ ὡς ἡ α δ ζ γωνία πρὸς τὴν α δ β, οὕτως ἡ ε ζ πρὸς τὴν ε β· καὶ ὡς ἄρα ἡ ε ζ πρὸς τὴν ε β, οὕτως ἡ α γ περιφέρεια πρὸς τὴν α β περιφέρειαν· ἐναλλὰξ ἄρα ἐστὶν ὡς ἡ ε ζ πρὸς τὴν α γ περιφέρειαν, οὕτως ἡ ε β πρὸς τὴν α β περιφέρειαν· μείζων δὲ ἡ ε ζ τῆς α γ περιφερείας· μείζων ἄρα καὶ ἡ ε β τῆς α β περιφερείας· ὅπερ ἔδει δεῖξαι.

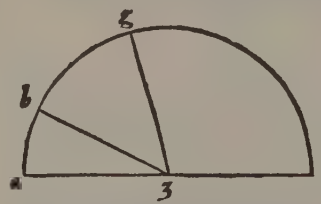
Has propositiones ita Latinè conuertimus.

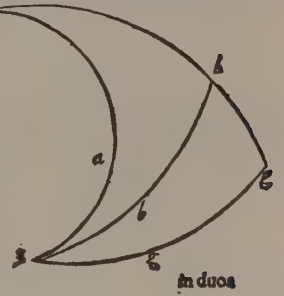
1 Si quis arcus notus in duos diuidatur, quorum sinuum ratio inter se constet, ipsi arcus etiam cognoscentur.

Ex hoc sequitur, si dentur tota circumferentia α γ, & ratio subtensæ duplo circumferentiæ α β ad subtensam duplo circumferentiæ β γ, dabitur quoq; vtraq; circumferentiarum α β & β γ. Cõstituta enim eadem descriptione, coniungãtur recta linea α δ & à signo δ agatur cathetus δ ζ in ipsam α ε γ. Cum igitur detur circumferentia α γ cum angulo α δ ζ semissem eiusdem circumferentiæ subtendente, etiam totum α δ ζ triangulum manifestum erit. Postquàm verò α γ arcum subtendens tota data est, & ex constituto ratio α ε ad ε γ, quę eadem rationi subtensæ duplo circumferentiæ α β ad subtendentem duplum β γ, etiam data erit α ε, & reliqua ζ ε. Igitur data δ ζ, dabitur etiam angulus sub ε δ ζ trigoni rectanguli ε δ ζ, & totus angulus, qui sub α δ β continetur. hinc etiam circumferentia α β & reliqua ε γ, id quod demonstrare oportebat. Praxis huius propositionis facilis sit: quæratur sinus versus dimidij arcus dati, qui auferatur ex sinu toto, & ex α ε auferatur media pars α γ, tum relinquẽtur ζ ε. Iam notis lateribus ζ ε & ζ δ, inuenietur angulus ζ δ ε, & iungatur α δ ζ angulo, vt habeatur totus α δ β.

XXIII.

Si angulum notum in duos particulares secueris angulos, quorũ sinus proportionem habeant datam, uterq; eorum notus erit.





Hæc ad angulos superficiales & planos rectilineos & sphærales accommodabitur. Sit angulus huiusmodi notus a 3 g diuisus in duos

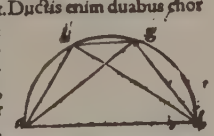
Plate 4. Regiomontanus, edited by Daniel Santbech, Basle, 1561.

æqualis, vnde triãgulus a b e, est æquiangulus triangulo d b g, igitur per quartam sexti latera erunt proportionalia. Erit igitur a b ad b d, sicut a e ad d g, & permutatim a b ad a e, sicut b d ad d g, ergo per decimam quintã sexti, quod fit ex a b in d g, est æquale ei quod fit ex a e in b d. Iam autem demonstratũ est, quod fit ex a d in b g, est æquale ei quod fit ex e g in b d. Igitur per primam secundi totum rectangulum, quod fit ex ductu a g in b d, æquatur duobus rectangulis, quorũ vnum fit ex a d in b g, & aliud ex a b in d g. Nã quod fit ex a g in b d, æquatur duobus rectangulis per primam secundi, scilicet vni quod fit ex b d in e g, & alij quod fit ex b d in a e, simul sumptis. Sed primũ rectangulũ æquatur ei quod fit ex a d in b g, & aliud ei quod fit ex a b in d g, vnde quod fit ex a g in b d, est æquale duobus rectangulis, scilicet ei quod fit ex a d in b g, & illi quod fit ex a b in d g, simul sumptis, quod est propositũ.

PROPOSITIO III.

Si in semicirculo chordæ arcuum inæqualium notæ fuerint, chorda quoq; arcus quo maior minorem superat, erit nota.

Sint in semicirculo a b g d supra diametro a d descripto, duæ chordæ a b & a g notę. Dico quòd chorda arcus b g nota erit. Ductis enim duabus chordis b d & g d, quæ cum duæ a b & a g sint notæ, erit manifestum per corollarium primæ huius, eo quòd quęlibet earum est chorda residui semicirculi. Est igitur quadrilaterum a b g d infra circulum, cuius duæ diametri a g & b d sunt notæ, tunc per præmissam duo recta simul, quæ fiunt ex a b in g d, & ex b g in a d nota erunt. Rectangulũ autẽ quod fit ex a b in g d, est notũ, eo quòd ambæ lineæ ipsum rectangulum continentes sint notę, quoabhinc de totali rectangulo, quod fit ex a g in b d, manebit rectãgulum quod fit ex b g in a d, & quia vna eius linearum ipsum continentiũ est nota, quia a d diameter circuli erit per diuisionẽ, reliqua linea scilicet b g nota, quod est propositum.



PROPOSITIO IIII.

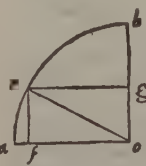
Si in semicirculo alicuius arcus chorda nota fuerit, chorda quoq; quæ eius medietati subtenditur, nota erit.

Si in semicirculo a b g descripto supra diametro a g, arcus b g chordam notam habens, diuiso arcu b g per æqua per 29 tertij, & ductis chordis a b, b d, a d & d g, ducatur perpendicularis d f supra diametrum per 12. primi. Dico quòd linea f g est medietas superfluj lineæ a g super lineam a b. Pono enim lineam d e æqualem lineæ a b per tertiã primi, & produco d e. Et quia a b est æqualis a e, posita a d communi, erunt duæ lineæ a b & a d trianguli a b d, æquales duabus lineis a e & a d triãguli a e d, quælibet videlicet suæ relatiuę, & arcus b d æqualis arcui d g, & per 26. tertij, angulus b a d æqualis angulo e a d, igitur per quartam primi basis b d æqualis basi e d. Et quia linea b d per 28. tertij, est æqualis lineæ d g. Igitur d g est æqualis d e, igitur per quintam primi, trianguli d e g anguli supra basim sunt æquales. Quare d f linea demissa per 26. primi, diuidit e g in æqualia. Tota autem e g est superfluum lineæ a g super a b, & f g est medietas superfluj, & ita patet quod dictum est. Et quia chorda arcus b g est nota ex hypothesi, erit chorda residui semicirculi, quæ est linea a b nota, quæ est æqualis a e, erit igitur e g nota, & per consequens eius medietas f g. Quia ergo per 30. tertij, an-

RECERE maiores nostri sinus & chordarũ tabulas, quorũ vsus maximè necessarius est, certas aliarum tabularũ numerationes reddere volenti. Verùm omnes illi diametrũ circuli paucarũ admodum partium cõstituerunt, veluti Ptolemæus 120, Arzahel 300, vnamquamq; partium in 60 minuta, minutumq; in 60 secunda distinguentes. In arcu etiam, tantùm per quartam gradus lineam numerũ in sinubus auxerũt, propter quod fit, vt cum ex arcu sinum, aut econtrà ex sinu arcũ elicere velimus, sæpe necesse sit sumere partes proportionales, itemq; in vsu sinuum, partes in minuta, minutaq; in partes reducere. Quod profectò nedum parum in arte numerandi instituto, sed etiam peritissimis tædium parit. Vt igitur hoc impedimẽtum tolleretur, facilisq; fieret sinum inuẽtio, conatus sum nouas tabulas fabricare, quarum extensio in arcu per singula minuta procederet ipsamq; circuli semidiametrũ, quæ sinus totus est, ne amplius aliqua subdiuisione opus esset, 6000000 partium fore supposui. Compositio verò ipsa talem habuit progressum.

PROPOSITIO I.

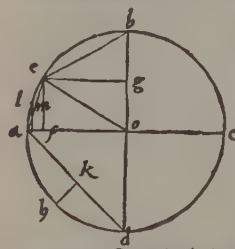
Cognito sinu alicuius arcus quarta circuli minoris, notus fiet, & sinus complementi talis arcus.



Nam quadratũ semidiametri æquale est duobus quadratis sinuum duorũ arcuum & sui cõplementi, vt in quarta a o b, arcus a e sinus sit e f. Arcus autem e b, sinus sit e g, quadratum e o, æquale est duobus quadratis linearum e f & f o, sed f o æqualis est e g &c.

PROPOSITIO II.

Sinus arcuum per Kardagas authorum ostendere.



Kardaga portio arcus 15 gr. appellatur. Pro huius ostensione sit circulus a b c d, super centro o, duabus diametris eius orthogonaliter sese secantibus a c, b d, arcus a e sit 30. grad. eritq; e b, 60 gr. propterea erit e b linea recta latus hexagoni circulo inscriptibilis, ideo æquale semidiametro e o, aut o b. Quare e g perpendicularis super o b, diuidet o b in partes æquales, sed e f sinus arcus a e, æqualis est & æquedistans o g. Ideo nota o b sinu toto, nota erit e f sinus arcus 30 gr. quia medietas sinus totius: hinc ex priore cognita fiet linea e g, quę sinus est portionis 60 gr. Præterea facta chorda a d, & arcu a h, 45 gr. h k diuidens a d per æqualia, distinguet a k sinum 45 gr. qui patebit ex hoc quòd quadratum semidiametri duplum sit quadrato lineæ a k. Deniq; ducta chorda a e, diuisaq; per mediũ in m, fiet a m sinus arcus 15 gr. qui innotescet ex quadratis a f & f e, ea enim coniuncta faciunt quadratũ a e, quod quadruplum est quadrato lineę a m. Tandem ex sinu arcus 15 gr. & propositione prima cognitus fiet sinus arcus 75 gr. Sic omnium arcuum per Kardagas authorum sinus patefacti sunt. Præsupposui autem in inuentione horũ sinuum propter maiorem præcisionem, semidiametrũ circuli partes habere 600000000. & secundum hoc repperi sinus arcuum illorum, vt hic habes.

Arcus

Plate 5. Regiomontanus, edited by Daniel Santbech, Basle, 1561.

directionum.[26] A few years later, in 1471, he established himself at Nuremberg, and became an intimate associate of Bernard Walther, a rich citizen and amateur astronomer, under whose patronage he was furnished with an observatory and a workshop for constructing astronomical instruments.[27]

Pope Sixtus, having proposed the reform of the calendar, desired that Regiomontanus should give his time and talent to the enterprise. In 1475 Regiomontanus returned to Rome to answer the call of the Pope, leaving his unfinished work to the care of Walther.[29]

In less than a year, death overtook him. It is conjectured that the son of George of Trebizond shared with his father a feeling of enmity for Regiomontanus and that it was he who caused the untimely death of the young scholar.[30] The Pantheon is said to be his resting place, although the tomb is unknown.[31]

3. WORKS OF REGIOMONTANUS

Regiomontanus shows a twofold purpose in his trigonometrical works: (1) he did for the West what Naṣir ad-Dīn had done two centuries previously for the East, namely, he constructed a uniform foundation and a systematic ordering of trigonometric knowledge and (2) he made observations for the bettering of tables for trigonometric calculations.

It will be in keeping with the object of this study to treat in detail only two of the works of Regiomontanus: De triangulis libri quinque, emphasizing the first purpose, and Tabulae directionum, the second.

a. De triangulis libri quinque

At the death of Regiomontanus, Walther purchased from the illiterate heirs of his scholarly protégé his manuscripts, books, documents, and instruments; he was the jealous guardian of these treasures until his death; then the Senate of Nuremberg bought this rich collection.[32] This somewhat accounts for the tardy appearance of the posthumous works of Regiomontanus. De triangulis libri quinque was printed at Nuremberg by John Petreus in 1533. A second edition was made in 1561 by Daniel Santbech at Basle. This trigonometry did not include the use of the tangent function and, in other ways, was not as advanced as the trigonometry of Naṣir ad-Dīn and other writers in Persia and Arabia.

A careful comparison of the two editions of De triangulis libri quinque reveals no striking dissimilarities. The 1533 edition has one hundred thirty-seven pages with many illustrations and diagrams. The 1561 edition has one hundred forty-eight pages; tables on nineteen unnumbered pages follow the text; this edition also has numerous illustrations and diagrams.

The first edition has a dedicatory epistle by Johann Schöner,[33] a letter to the reader by Regiomontanus, and five books on trigonometry. The second edition has the same except for the following additions: Tractatus of George Peurbach, Compositio tabularum sinuum rectorum of Regiomontanus, tables for the radius equal to 600 000 000 of the 96 values of the sine that can be readily obtained from the propositions in the

26. A. von Braunmühl, loc. cit.
27. F. Boquet, op. cit., p. 237.
28. Idem.
29. Idem.
30. Pierre Gassendi, op. cit., p. 470.
31. Idem.
32. F. Boquet, op. cit., p. 240.
33. The geographer, astronomer, and geologist, Johann Schöner, was born at Carlstadt in 1477 and died at Nuremberg in 1547. He received his education at Nuremberg and Erfurt; taught at Bamberg; and finally obtained the Chair of Mathematics at the Gymnasium of Nuremberg (Boquet, op. cit., p. 245).

Compositio, tables of sines for the radius equal to 6 000 000 for the interval of one minute, and tables for the radius equal to 10 000 000 for the same interval.

The second edition has some notes from Ptolemy given in both Greek and Latin in connection with Theorems 22, 23, 24 of Book IV, which do not occur in the first edition.

The first edition lacks the figure for Proposition 23 of Book II. The second figure for the eighteenth proposition in Book IV of the second edition differs slightly from that of the first edition.

The following extract from the preface of Regiomontanus, indicates the devotion of Regiomontanus to his teacher, George Peurbach, who had confided to him the completion of a work which Peurbach himself could not complete on account of his failing health:

> Although I have written these books on triangles after the Epitome, using a reverse order, [i.e.,] composing the introductory material at a later time,[34] (and I pass on this method), yet no one will adequately understand our triangles through the medium of the stars. But if there is any undue criticism with regard to my procedure, unless I am mistaken, that critic is rightfully silent when he considers how good and just it is to comply with the wishes of one's elders. Since the health of my teacher was failing, it was necessary to comply with his command; he, having recently finished six books on the heavenly bodies, left the remaining seven to his Johann to finish as soon as possible. So much did [Peurbach] value the counsels of Bessarion that he dedicated to that most worthy leader what was thus far finished and what he, already near to death, tried to complete. Therefore, having complied with the commands of my teacher, many things have come to my mind by making use of plane and spherical triangles, which repeatedly occurred in the first six books of George, and influenced me to write this treatise on triangles....

There are twenty-three items under the caption "Definitions" in Book I; no other definitions are given in the text. Some of the definitions deal with magnitudes or quantities. For instance:

> Any quantity is said to measure another quantity when it is contained in the other according to a known number or is found to contain the other according to the units of a given number.
>
> The difference of two unequal quantities is that portion of the larger by which it exceeds the smaller one.

Others pertain to ratios:

> We will say a ratio is given when either its measure is given, or a ratio equal to it has known terms.
>
> Equal ratios are those which have a common measure (or denomination by the same number).

A few are concerned more directly with geometry and trigonometry:

> An arc is the part of the circumference of a circle.
>
> An arc and a chord having been bisected, we call the half of the chord of the bisected arc, the 'sinus rectus.'
>
> The complement of any arc whatever is considered to be the difference between itself and the quadrant.
>
> The complement is defined in another way as the measurement of the difference between the angle and a right angle.

The "Communes animi conceptiones" of De triangulis libri quinque with the exception of the last one bear no little resemblance to those of Euclid's Elements. For exple:

34. It was customary for an astronomical treatise to have a trigonometric introduction. The Epitome to which Regiomontanus refers is the translation and commentary of Ptolemy's Almagest which was begun by Peurbach. The "traditional trigonometric introduction" was written after the astronomical work itself and is the De triangulis libri quinque.

Equal magnitudes are measured in the same manner.

The same measure is contained equally in two or in any number whatever of equal magnitudes.

If you take away equals from two unequal magnitudes, the remainders will have the same difference as did the whole magnitudes.

And if from equal magnitudes you subtract unequal portions, then the remaining quantities and the subtracted quantities, taken in reverse order, yield equal differences.

All ratios can be given in numbers.

Of the fifty-seven propositions of Book I, the first nineteen treat of mangitudes and ratios, the remaining thirty-eight deal geometrically with the solutions of right, isosceles, and scalene triangles. Each proposition has, as a general rule, an accompanying figure and a numerical illustration.

Oddly enough Proposition 5 is an elaboration of one of the common notions: "Every two magnitudes given together have the same ratio as that of the two numbers according to which they themselves are measured. Whence we proclaim as obvious, all given ratios can be reduced to numbers."

If a and b are two given magnitudes, by the definition of a common measure, e measures a according to the number e; b, according to f. I say that the ratio a to b is the ratio of the number e to f, for if d measures a according to the number e, d will be, by definition, contained in a as many times as there are units in the number e; hence a to d is the same ratio as e to unity. Similarly, d is in b as many times as there are units in the number f, wherefore the ratio d to b ias as unity to f. Therefore it pleases us to conclude that the number e to the number f is the same kind of a ratio as the quantity a to the quantity b.

Proposition 19 is the explicit statement of the rule of four quantities: "If of four proportional magnitudes, any three will be given, the remaining fourth one will become known."

Proposition 20[35] is concerned with the sine function: "In every right triangle if you describe a circle, radius equal to the longest side, and center, the vertex of one of the acute angles, the 'sinus rectus' of the arc belonging to this angle will be the side subtending this acute angle; but the sine complement of this arc is equal to the third side."

Let abc be a right triangle having c as a right angle. The center and radius of the required circle being a and ab respectively; the circumference of the circle intersects ac, prolonged, at c I say that the side bc opposite the angle bac is the sine of the arc be subtending this angle. The third side ac, will evidently be equal to the sine complement of this angle. Extend bc so as to intersect the circumference at the point d. Since a is the center of the circle, ak may be drawn parallel to bc; the chord bh is parallel to the side ac; bh and ak necessarily intersect at g, the angles abh and bak are equal to the acute angles of the given right triangle. The semi-diameter ac is perpendicular to the chord bd, because angle acb is a right angle (Book III, 33 of the Elements); according to 29 of this same book, the arc bd is bisected. The whole line bd is, by definition, the chord of the arc bd; whence half of it, i.e., bc, is the sine of half the arc be subtending angle bae

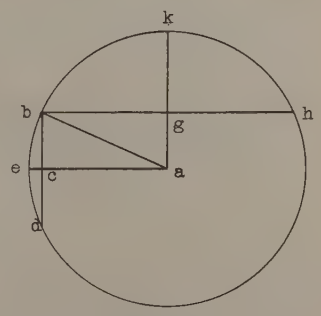


Fig. 12

35. This porposition without the proof is given in A. von Braunmühl, op. cit., p. 125, and J. D. Bond, op. cit., p. 321. However, contrary to Bond's statement this proposition is not the only one in Book I that mentions the sine, for the sine function is employed in the proofs of two other propositions which will be considered in this chapter.

or bac, which is what the first part of our theorem asserts. From Book I, 34 of the Elements, agb is a right angle; considering the semidiameter ak, the chord bh, the arc bh, and what has been stated above, bg will be the sine of the arc bk; bg is also equal to the side ac of the triangle abc. Because agbc is contained in parallel lines, the angle cag or eak is a right angle by Book I, 29 of the Elements, ek is a quadrant and bk is defined as the complement of be whose sine bg is equal to ac. Therefore both parts have been sufficiently demonstrated.

Proposition 27 likewise makes use of the sine function: "To determine all the angles when two sides of a right triangle are known."

Let the triangle abc have the right angle at c and let the two sides ab and ac be known. I say that all the angles will also be known. On the vertex of angle b (which subtends one of the given sides) as center describe a circle with radius equal to the length of the side ba. According to Proposition 20, ac will be the sine of the angle abc. By hypothesis, the two lines ab and ac are given according to a certain measure; but ab is the semidiameter of a circle and can be given according to a new measure, namely, a unit part of the 'sinus totus.' According to Proposition 7[36] ac will also be given by this measure; hence ab is the 'sinus totus' or 'sinus quadrantis' and ac is the known sine. By the table of sines, which if neglected, prevents us from finding out anything in this proposition, we learn its arc, thus knowing the arc of the sine, the angle which replaces that arc, for the arc itself is known. Angle bac is known by Proposition 21[37] and angle bac by Proposition 22.[38] Therefore all our angles are known which is what we desired to show.

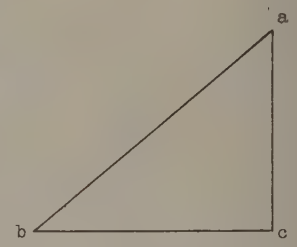


Fig. 13

Operation:

The length of the side subtending the right angle is first, the length of the other known side is second, and the total sine is third. Multiply the second by the third and divide the product by the first; this is the sine of the arc of the angle that is being sought. From the table of sines, the numerical value of the angle can be found. The difference between this angle and the right angle will be the numerical value of the other acute angle.

Let ab equal 20, ac equal 12, then bc is 16. Let us suppose the tables to be calculated with the 'sinus totus' equal to 60 000. Multiply 12 by 60 000, this equals 720 000. Divide by 20, this equals 36 000 which is the sine of the angle 36° 52'. The other acute angle will be 53° 8'.

$$\frac{\sin a}{\text{total sine}} = \frac{12}{20}$$

The sine function is also employed in the proof of Proposition 28: "To inquire about the angles when the ratio of two sides of a right triangle is known."

One of the two sides whose ratio is known is either opposite the right angle or not. Consider the first alternative. Let the side ab be opposite the right angle acb, let the ratio of the side ab to ac be known. I say that the angles of this triangle will also be known. ac is the sine of the arc of the angle abc; ab, the semidiameter of the circle, is evidently the 'sinus totus.' Hence the ratio of the 'sinus totus' to the sine of the angle abc is known; wherefore the sine of the angle can be determined, and finally the angle itself will be known.

If, however, the ratio of the two sides bc and ac is known,[39] the ratio of their squares will also be known; by adding the terms of this ratio, a term corresponding to the square of ab will be obtained, because angle c is a right angle. Hence the ratio ab to ac is known and angle bac can be found from the above case.

36. "If two magnitudes are given, one of which is known by a new measure, then the remaining one becomes known through this same measure."
37. "It is certain that every right angle is known."
38. "If one of the two acute angles which every right triangle has is known, the remaining one is not unknown."
39. Regiomontanus' unfamiliarity with the tangent function at the time he wrote the De triangulis libri quinque, is evidenced by his procedure in this case. Compare with A. von Baunmühl, loc. cit. See p. 35, where the "tabula fecunda" (table of tangents) is considered.

Theorem 29 is the only one in Book I, outside of Proposition 27 already considered, that pertains to the solution of a right triangle. It is: "If I know one or the other of the acute angles which a right triangle has and also one of its sides, I can measure the other angle and the sides."

The other propositions of Book I, relating to triangles, have, from our point of view as explained below, a triple classification: (1) those which time has changed into "exercises for the student"; (2) propositions relating to the four cases of the solution of oblique triangles; and (3) theorems which no longer have place in trigonometric texts.

The following propositions belong to the first group:

Proposition 30: If either of the acute angles of a right triangle are given, we will know the ratio of the sides, although the sides are not given.

Proposition 32: In every triangle consisting of three known sides, if one of the perpendiculars is known, the remaining two will be known.

Proposition 37: It is easy to investigate the angles of an isosceles triangle from a knowledge of the sides and base.

Proposition 38: If any side of an isosceles triangle is given, or if a perpendicular with one of the angles is given, the measurements of the remaining sides and perpendiculars can be determined.

Proposition 39: Knowing the sides of an isosceles triangle with the base, determine all of its angles.

Proposition 48: Knowing two angles of a triangle, find out by turns the ratio of the sides.

Proposition 54: The ratio of three sides of a triangle being given, determine its angles.

Proposition 55: Three angles of any kind of a triangle being given, find the ratio of the sides.

Proposition 56: The ratio of two sides being given and knowing one of the angles, the remaining two angles will be known.

Proposition 57: The ratio of two angles each (severally) being given to a right angle, and knowing one of the sides, all the angles and the remaining sides will be known.

The theorems corresponding to the four cases of the solution of oblique triangles are:

Proposition 52: If a given side of a triangle forms two known angles, the two remaining sides will not be unknown.

Proposition 53: The known side of a triangle which is opposite either of two given angles will make known the remaining sides.

Proposition 50: If either of two known sides of a triangle be placed opposite a given obtuse angle, a Geometer will not be ignorant of either the remaining angles or the remaining side.

Proposition 49: If two sides of a given triangle include a known angle, give the measure of the remaining angles and the remaining side.

Regiomontanus has a series of propositions leading up to the theorem corresponding to the case of "three given sides." They are:

Proposition 42: "A triangle has three known sides, investigate the size of any of its angles."

Here Regiomontanus uses the Pythagorean theorem and shows how to determine whether the angles are acute, right, or obtuse.

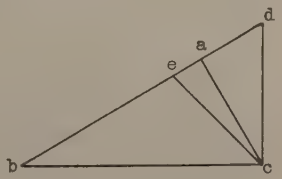


Fig. 14

Proposition 43: "Three sides of a triangle being given, ascertain the two portions of the base made by the perpendicular drawn from the angular point opposite the base."

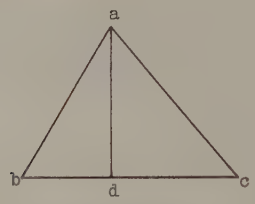


Fig. 15(a)

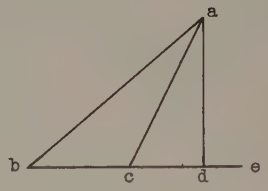


Fig. 15(b)

Here Regiomontanus implicity uses the formular wherein one portion of the base is equal to $(\overline{ac}^2 + \overline{bc}^2 - \overline{ab}^2)/2\ bc$.

Proposition 44: "Investigate in another way what the preceding has shown."

The formula employed in this case is: One portion of the base is equal to $\dfrac{bc - (\overline{ab}^2 - \overline{ac}^2)/bc}{2}$

Proposition 45: "To consider the portions by still another way."

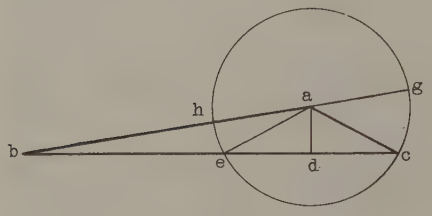


Fig. 16(a)

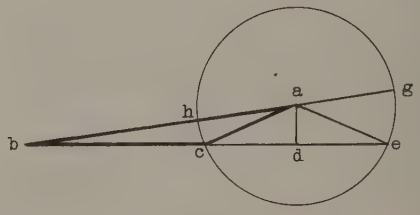


Fig. 16(b)

The formula Regiomontanus uses here is: One portion of the base is equal to $\dfrac{bc - (ab + ac)(ab - ac)/bc}{2}$.

An exercise accompanying Proposition 46 shows that this proposition is concerned with the altitude of a triangle. This exercise is: "The square of either of two portions (of the base) subtracted from the square of the side intersecting the portion equals the square of the perpendicular (altitude)."

Proposition 47: "Anyone knowing three sides of a triangle will also know its angles." The proof is based on the proposition concerning a perpendicular drawn from an angular point to the side opposite the angle.

Those theorems which might be considered in the third group are:

Proposition 25: Two angles of any kind of a triangle being known, the third angle will also be known.

Proposition 26: Two sides of every right triangle being known, the third will also be known.

Proposition 31: If upon the base of a triangle, a perpendicular is let fall from the vertex of whatever kind of an angle has been constructed, then if it cuts the base outside the triangle, one of the two angles must be obtuse; if the straight line coincides with the side, one of the angles will be right, but it will be understood to be acute if it falls within the triangle.

Proposition 33: We will show that in every equilateral triangle the three angles are known, whence it will be established that each of them is acute.

Proposition 34: The square of the side of every quilateral triangle is four-thirds times the square of the perpendicular. Hence, again from the known side, we have a knowledge of the perpendicular, and conversely.

Proposition 35: If I know any angle whatever of an isosceles triangle, the remaining angles will be known.

Proposition 36: It is easy to find the perpendicular coterminal with the equal sides of an isosceles triangle which divides the known base into two equal parts.

Proposition 40: If you have the perpendicular of an isosceles triangle given, find out the side from the known base, or conversely the base from the known side.

The Sine Law is the first proposition of Book II. Its accompanying figure and proof bear a close resemblance to those of Al-Bīrūnī.[40] "In every right lined triangle the ratio of a side to a side is as the 'sinus rectus' of the angle opposite one of these sides to the 'sinus rectus' of the angle opposite the other."

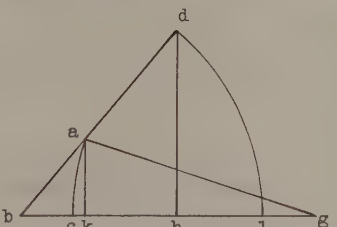


Fig. 17

Of the thirty-three propositions of Book II, there are two which have algebraic solutions. Proposition 12 is one of these: "Given the perpendicular, base, and ratio of the two sides, find the magnitude of both sides."

Let the side ab be to ag as 3 is to 5 (ab < ag); the base bg, 20 ft.; the perpendicular ad, 5; on bg take de equal to bd.

If eg equals 2 "res," then the line be will be 20 minus 2 "res," and half of it will be 10 minus 1 "res." I multiply bd by itself; this will produce 1 "census" and 100 minus 20 "res," to which I add the square of the perpendicular which is evidently 25; taking everything together, I have 1 "census," 20 "res," and 125; thus we have the ratio of the squares of two lines equal to the ratio 9 to 25. The result is 16 "census" and 2000 equal 680 "res." Hence eg which is equal to 2 "res" can be found.

In modern notation this is:

$$\frac{X^2 - 20X + 125}{X^2 + 20X + 125} = \frac{9}{25},$$

Hence:

$$16X^2 + 2000 = 680X.$$

The proof of Proposition 25 is given by both von Braunmühl[41] and Bond.[42] "If we accept as known the base of a triangle with the perpendicular or area of the triangle and the angle which the base subtends, we can give an estimate of both the sides."

Proposition 26 is the first implicit statement of the trigonometric formula for the area of a triangle. In this proposition Regiomontanus demonstrates that if the area and the product of two sides of the triangle are given, then either the angle included by the two sides or its supplementary angle will be known. Both von Braunmühl[43] and Bond[44] state this proposition but do not prove it.[45] The proof is as follows:

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40. See p. 9.
41. Op. cit., p. 126.
42. Op. cit., p. 322.
43. Op. cit., p. 127. "Den der Basis des Dreieckes gegenüberliegenden Winkel zu berechnen, wenn das Produkt der ihn einschliessenden Seiten und die Flache des Dreieckes bekannt sind."
44. Loc. cit. "The area of a triangle and the product of two sides being known, either the included angle or its supplement will emerge."
45. A translation of both proof and proposition is given by Vera Sandford in D. E. Smith, A Source Book in Mathematics (New York: McGraw-Hill, 1929), pp. 431-433.

Using the diagram of the preceding propositions, if the perpendicular bk drawn to the line ag falls outside the triangle, then the ratio of bk to ba will be known;[46] by Proposition 28 of Book I,[47] we accept angle bak as known; angle bag is known, because it is the supplement of bak. But if the perpendicular bk falls inside the triangle as is seen in the third diagram (Fig. 18) of the preceding proposition, then as before, the ratio bk to ba will be known, and we conclude that angle bak or bag is known. But if the perpendicular bk coincides with the line ab, angle bag is a right angle and therefore is known; this occurs because the area of the proposed triangle equals [half] the product of the two sides.

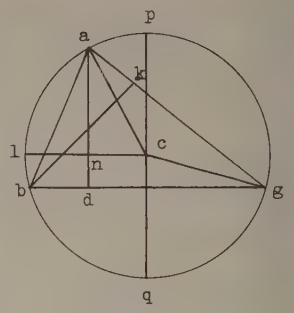

Fig. 18

Proposition 24 is rather interesting. The figure is closely related to the one that is used in the proof of the "Theorem of Ptolemy." The proposition is: "Given three sides of a triangle find the diameter of the circle circumscribing it."

Regiomontanus constructs a perpendicular dg to the line bg in the triangle abg. From b he constructs the perpendicular bz to ag. With bg as one side of the angle he constructs at b an angle equal to angle abz. The side of this angle will intersect the perpendicular to g at d. The required diameter is bd.

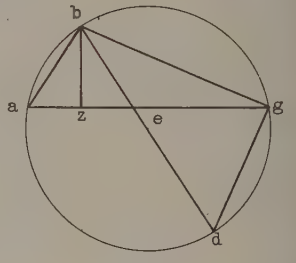

Fig. 19

There are only two propositions in Book II which pertain to the solution of oblique triangles; they are:

Proposition 4: If anyone will give two angles of any kind of a triangle with any one of its sides, I will easily measure the remaining sides.

Proposition 5: Two sides of a triangle with an angle opposite one of them being given, ascertain the remaining angle and the third side.

There are nine propositions dealing with the sum and difference of sides. For example:

Proposition 2: Knowing the sum of two sides of a triangle with the two angles opposite the sides, distinguish the sides of the triangle one from the other.

Proposition 14: If the lengths of the two portions[48] of the base are given as unequal and the sum of the sides is given, find the magnitude of the sides.

Proposition 15: The base of the triangle which subtends a known angle being given, and the sum of the two sides being known, show a way of measuring both the sides and the angles opposite them.

Proposition 20: If anyone will give the difference of the sides, the difference of the portions of the base, and the angle which subtends the base, he will receive a knowledge of all the sides.

Propositions 54 and 55 of Book I are duplicated in Book II by:

Proposition 6: A triangle with three known angles will have the ratio of its sides known.

Proposition 9: From the ratio of three sides of a triangle give the measurement of all its angles.

---

46. Let the area of the triangle abg = A and let ab.ag = B, then ag = B/ab. Since bk is perpendicular to ag, A also equals $\frac{1}{2}$ bk.ag, so ag = 2A/bk; therefore bk/ab = 2A/B.
47. See p. 22.
48. Regiomontanus uses the term "casus" for the portions of the base made by the perpendicular drawn to it from the opposite vertex. This expression is also found in the work of Maurolyco. See p. 67.

The remaining propositions of Book II would be classed as "exercises" in a modern trigonometry text. A few of these theorems are:

Proposition 7: Given the perimeter of a triangle with two of its angles, find out each one of the sides separately.

Proposition 31 might be chosen as an illustration of these exercises: "Two triangles having two equal and known sides are drawn upon one base, investigate the distance of one vertex from the other." Here Regiomontanus constructs the two triangles abc and dbc on the known base bc. Because another side of each triangle is known, it is possible to measure the lengths of the perpendiculars ae and dg; if they are equal, then eg is the distance between the vertices; if they are not equal, it is possible by means of the Pythagorean theorem to measure hd and hence ad.

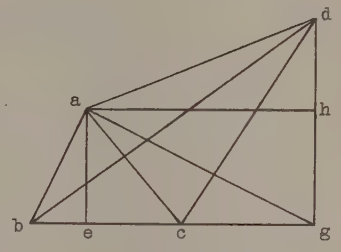

Fig. 20

The fifty-six theorems of Book III show some originality; certain sections resemble the Spherics of Menelaus, while some of the figures have a similarity to those in the work of Geber.[49] No proposition employs a trigonometric function. This book of the De triangulis libri quinque is more on the order of a spherical geometry.

A few theorems will illustrate the contents of this book:

Proposition 1: If a sphere is cut by a plane, the common part of the sphere and the intersecting plane will be the circumference of a circle. Thus it will be established that the foot of the perpendicular going from the center of the sphere to the intersecting surface is the center of the circle.

Proposition 3: A straight line which proceeds orthogonally from a center of a circle placed on a sphere will necessarily contain the center of the sphere.

Proposition 5: Every straight line from the center of a circle on a sphere, if drawn orthogonally will pass through the poles of the circle.

Proposition 16: A great circle intersecting one of the poles of another circle, will also pass through the remaining pole.

Proposition 35: In any two spherical triangles where the sides of one are equal to the sides of the other, the angles opposite these sides will also be equal.

Proposition 42: In every spherical triangle, the greater angle subtends the greater side.

Proposition 55: In any two spherical triangles where two sides of one are equal to two sides of one are equal to two sides of another and an angle of one is equal to an angle of the other, if the triangles are to be equilateral and equiangular, it is necessary that both the other angles be either acute or obtuse.

Regiomontanus' Book IV bears a close resemblance to Geber's Book III.[50] A number of the theorems are the same type as Proposition 12 which states: "If any triangle has two unequal acute angles, the side opposite the smaller angle will be less than a quadrant."

In triangle abg, the two angles b and g are acute and angle g is less than angle b. I say that ab is less than a quadrant. At g and b, construct two right angles and produce arcs bd and gd meeting at d which together form a semicircumference because each is a quadrant; d is the pole of the circle bg. ba and ag are less than a semicircumference; arc ba is less than ag by Book

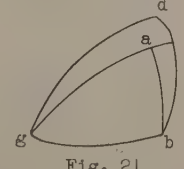

Fig. 21

49. A. von Braunmuhl, op. cit., p. 127.
50. Idem.

III, 42[51] because angle g is less than angle b. Therefore ba is less than a quadrant, which is what we wish to demonstrate.

In Proposition 16 and 17,[52] Regiomontanus states the Sine Law for a right triangle and then for all triangles: "In every right triangle, the sine of every side to the sine of the angle it subtends has the same ratio." "In every triangle which is not a right triangle, the sines of the sides to the sines of the angles opposite them have the same ratio."

In these propositions, Regiomontanus employs very lengthy proofs. In Proposition 16 he treats of the three possible cases wherein the two angles, other than the right angle, might both be right angles, one might be a right angle, or neither be a right angle. In the first case he recognizes the fact that the triangle is a polar triangle, even though he doesn't use that term. For the third case, he makes further subdivisions of the theorem, considering whether both angles are acute, obtuse, or one acute and the other obtuse. In Proposition 17, he divides the triangle into two right triangles and applies the conclusions of Proposition 16.

Proposition 18 is the same as "Geber's Theorem": "In every triangle having one right angle, the ratio of the sine of an angle which is not a right angle to the sine of the right angle is as the sine complement of the remaining angle to the sine complement of the side which it subtends."

Some of the propositions of Book IV bear a close resemblance to the rules of Book V in Naṣir ad-Dīn's treatise:

Treatise on the quadrilateral

Book V, Chapter III.

Rules:

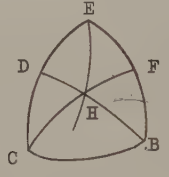

Fig. 22

(ζ) Every triangle which has three acute angles will have its sides less than a quadrant; its poles will fall outside the triangle.

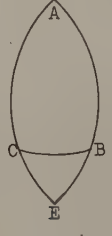

Fig. 24

(η) Every triangle which has an acute angle and two obtuse angles, has the side opposite these angles last mentioned greater than a quadrant, and the side opposite the acute angle less than a quadrant.

De triangulis libri quinque

Book IV.

Propositions:

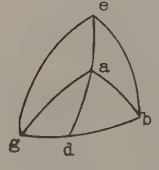

Fig. 23

9. Any triangle having three acute angles has its sides less than a quadrant.

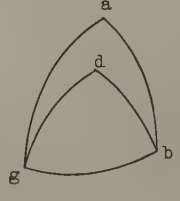

Fig. 25

11. Triangles having two equal obtuse angles will have each of the subtending sides greater than a quadrant.

51. See 27.

52. A translation of the proofs of these theorems by Eva Sanford can be found in D. E. Smith, op. cit., pp. 427-431.

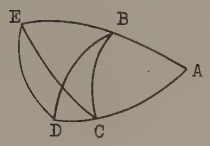

Fig. 26

(δ) Every triangle which has a right angle and two acute angles has no side greater than a quadrant...

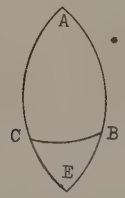

Fig. 28

(ε) Every triangle which has a right angle and two obtuse angles has the sides opposite the obtuse angles greater than a quadrant, and the side opposite the right angle less than a quadrant...

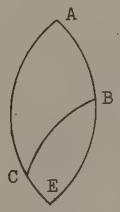

Fig. 29

(ϛ) Every triangle which has a right angle, obtuse angle, and acute angle will have the side opposite the acute angle less than a quadrant, and the two other sides greater than a quadrant...

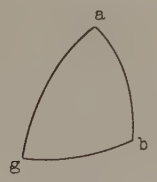

Fig. 27

7. If the side of a spherical triangle subtending a right angle is a quadrant, one of the other angles is also right; if less than a quadrant, both of the other angles are either acute or obtuse, and if the side is greater than a quadrant, one of the angles is obtuse while the remaining one is acute.

There are three propositions in Book IV which deal with the solution of right-angles spherical triangles. These propositions are:

Proposition 25: In every triangle which has a right angle and two known sides, the remaining side and angle will not be unknown.

Proposition 26: Three angles of a right triangle being known, all its sides will be determined.

In the proof of the above proposition, Regiomontanus uses the following formula:

$$\frac{\text{sine angle bag}}{\text{sine right angle abg}} = \frac{\text{sine complement angle agb}}{\text{sine complement side ab}} .$$

Proposition 27: One side and any angle which is not a right angle of a right-angled triangle is known; determine the remaining angle and the remaining sides.

The fourth book also gives the six cases for the solution of oblique spherical triangles. The six propositions are:

Proposition 28: Given two sides of a triangle which does not have a right angle, and also the included angle, the remaining side and angles will be known.

Because of a later reference to Proposition 29,[53] the figure and an explanatory note will be given here.

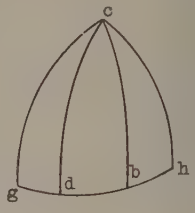

Fig. 30

"Knowing two sides of a triangle which does not have a right angle and an angle opposite one of sides, it is easy to display the knowledge of the remaining side and angle."

Here Regiomontanus considers the two possible cases that arise in this proposition. In a modern trigonometric text, this is known as an "ambiguous case."

Proposition 31: If any triangle not having a right angle, has two angles with the included side given, the remaining angle and sides will be known.

Proposition 32: Knowing two angles and a side subtending one of the angles of a triangle that does not have a right angle, the remaining angle and sides will be known.

Here, as in Proposition 29, Regiomontanus recognizes the possibility in the number of solutions.

Proposition 33: Given three angles of a spherical triangle not having a right angle; ascertain its three sides.

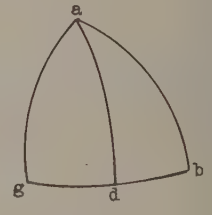

Fig. 31

In the proof of this proposition, Regiomontanus considers the oblique triangle abg; ad forms a right angle with gb; hence the following proportion is true:

$$\frac{\text{sine bad}}{\text{sine dag}} = \frac{\text{sine complement abg}}{\text{sine complement agb}}$$

Proposition 34: In any triangle, if three sides are known, the three angles will also be known.

Another interesting problem of this book might also be stated; it is Proposition 21: "If any known arc less than a semicircumference is divided in two, if the ratio of the sines of the segments is given, both of the segments will also be given."

The fifteen propositions of Book V, like those of the two previous books, deal with spherical triangles. In Proposition 2, which is the implicit statement of the Cosine Theorem for a spherical triangle, Regiomontanus uses a new function, "sinus versus" without previously defining it. "In every spherical triangle, the ratio of the 'sinus versus' of an angle to the difference of the 'sinus versus' of the side subtending the angle and the 'sinus versus' of the difference of the two arcs forming the angle is as the ratio of the square of the 'sinus totus' to the product of the sines of the arcs forming the angle."

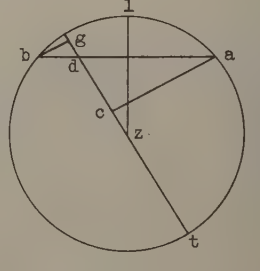

Fig. 32

Using the modern notation, this is what the theorem states.

$$\frac{\text{vers sin A}}{\text{vers sin a - vers (b - c)}} = \frac{1}{\text{sin b sinc c}}$$

53. See p. 50.

which can be reduced to the modern form of:[54]

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}.$$

This is the first time this rule was formed in a manner useful for calculation.[55]

The function "sinus versus" appears again in Proposition 11 where its use is more readily demonstrated: "Knowing the ratio of the 'sinus versus' of an arc to the 'sinus rectus' of the same arc, determine the arc."

> In the circle abgc having the diameter ag, and the chord bc bisected by the diameter at e, by definition ae is the 'sinus versus' of the arc ab and be is the 'sinus rectus' of this same arc; the ratio of ae to be is given. I say that it is possible to determine the arc ab. Draw the chords ab and bg, then according to Book III, 30, and Book VI, 8, of the Elements, the two partial triangles abe and ebg are both similar to triangle abg; from the corollary to the proposition last mentioned, be is the mean proportional between ge and ea. Since ea to be is known, then ge to ae will be known, whence ga to ae will be known.[56] But ga is the diameter, that is the chord of half the circumference; wherefore it is possible to determine the 'sinus versus' ae, whence arc ab will also be determined.[57]

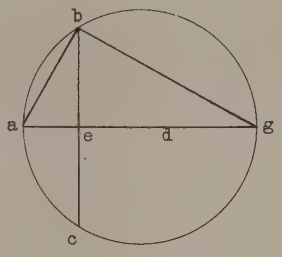

Fig. 33

As has been stated previously, the trigonometric portions of the two editions of De triangulis libri quinque have no striking dissimilarities, however the additions to the 1561 edition are not without merit; their consideration deserves place in this study.

It should be recalled that George Peurbach devoted many years of a fruitful study to the Almagest of Ptolemy. His Tractatus, a treatise on the proposition of Ptolemy concerning sines and chords, briefly illustrates the work of those studious years. The first part of the Tractatus ends with, "Hacc de mente Arzahelis," to indicate Peurbach's source for his methods of computation.

The six propositions of the second part of this treatise are from the Almagest.

> Proposition 1: A diameter of a circle being given, find the sides of a decagon, pentagon, tetragon, and equilateral triangle of all the circumscribed circles by the same circle. bd is the side of the hexagon; fd, the decagon; fb, pentagon; gb, tetragon; and al, equilateral triangle.

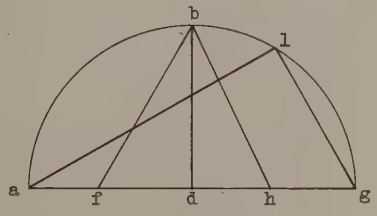

Fig. 34

54. Recalling that vers sine A equals 1 - cos A, and using a unit sphere, the formula becomes:

$$\frac{1 - \cos A}{1 - \cos a - 1 + \cos (b - c)} = \frac{1}{\sin b \sin c}$$

$$1 - \cos A = \frac{-\cos a + \cos (b - c)}{\sin b \sin c}$$

$$\cos A = \frac{\sin b \sin c + \cos a - \cos b \cos c - \sin b \sin c}{\sin b \sin c}$$

55. A. von Braunmühl, op. cit., p. 130.

56. ae/be = be/ge, hence ge/ac = $(be/ae)^2 = k^2$, a known value. (ge/ea) + 1 = (ge + ea)/ae = ga/ae = $k^2 + 1$.

57. ae = diameter/$(k^2 - 1)$; but ae = vers sin arc ab, so 1 - cos arc ab = diameter/$(k^2 - 1)$ whence cos arc ab = $\frac{(k^2 - 1) - \text{diameter}}{\text{diameter}}$.

Proposition 2: If a quadrilateral is inscribed in a circle, the product of the two diagonals is equal to the sum of the products of the opposite sides.

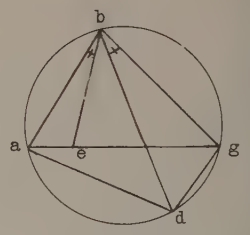

Fig. 35

Proposition 3: If, in a semicircle, chords of unequal arcs are known, the chord of the arc by which the larger exceeds the smaller will be known.

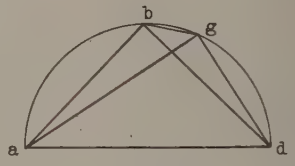

Fig. 36

Proposition 4: If in a semicircle, a chord of any arc is known, the chord which subtends half that arc will also be known.

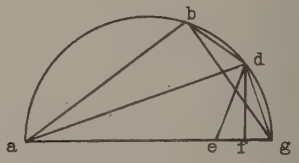

Fig. 37

Proposition 5: If two chords of two arcs in a semicircle are known, the chord which subtends the whole arc composed of these two arcs will also be known.

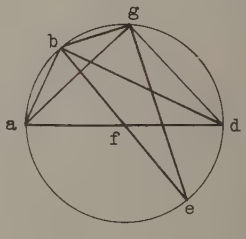

Fig. 38

Proposition 6: If two unequal lines are drawn in a circle, the ratio of the longer chord to the shorter will be less than the ratio of the longer arc to the shorter arc.

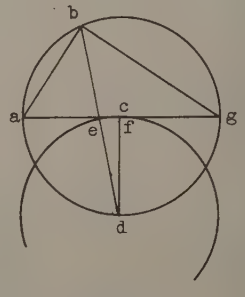

Fig. 39

The Compositio tabularum sinuum rectorum of Regiomontanus, together with the preceding treatise, was published separately by Johann Petreus in 1541 at Nuremberg. The contents of the Compositio is as follows:

Proposition 1: "Knowing the sine of any arc less than a quadrant find the sine of its complement."

The square of the semidiameter is equal to the squares of the sine of its arc and of its complement. In the quadrant aob, the sine of arc ae is ef; the sine of arc eb is eg; the square of eo equals the squares of ef and fo, but fo equals eg; hence knowing eg, it is possible to compute fo.

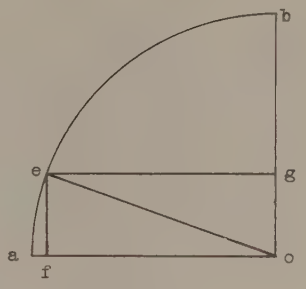

Fig. 40

Proposition 2: "To find the sine of an arc by kardagas."

An arc of 15 degrees is called a kardaga. For this demonstration let the circle abcd, center c, have two orthogonal diameters, ac and bd. Let ae be 30 degrees, then eb is 60 degrees. eb will be the side of an inscribed hexagon. eg is perpendicular to ob and bisects ob; ef is the sine of the arc ae and is equal and parallel to og. Since ob, the 'sinus totus' is known, ef, the sine of 30 degrees, will be known, because it is half of the 'sinus totus'; eg is the sine of 60 degrees. Draw the chord ad and make ah equal to 45 degrees; hk bisects ad; the square of the radius is twice the square of ak, which is the sine of 45 degrees. Divide the chord ae at m; then am is the sine of an arc of 15 degrees, which is known from the square of af and ef. From the sine of 15 degrees and the previous proposition, the sine of 75 degrees will be known. So the sines of all our arcs can be determined by kardagas. Let the semidiameter be equal to 600 000 000 parts and determine the sines of these arcs.

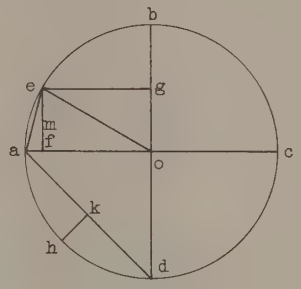

Fig. 41

Arcus	Sinus
90	600 000 000
30	300 000 000
60	519 615 242
45	424 264 069
15	155 291 427
75	579 555 496

Proposition 3: "The sine of an arc less than 90 degrees is the mean proportional between half the semidiameter and 'sinus versus' of twice the arc."

In the quadrant boc, the arc cr is known, cs is twice cr. Draw the lines sv orthogonal to oc; draw or intersecting cs at t; let x be the midpoint of oc. I say that ct is the mean proportional between xc and cv. Triangles oct and scv are similar; hence oc to sc is the same ratio as tc to cv; but oc to sc is as xc to ct; hence cv is to ct as ct is to xc.

Propositions 4 and 5 pertain to the construction of a regular pentagon, decagon, and hexagon in a circle. Proposition 6 is: "Ordinates of equal arcs are farther apart toward the center of a circle."

The following is a portion of the "tables for the radius equal to 600 000 000 of the ninety-six values of the

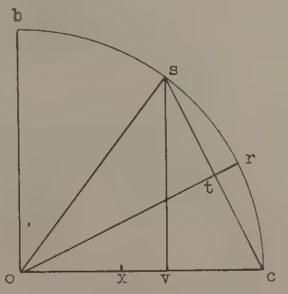

Fig. 42

sine that can be readily obtained from the properties in the Compositio." These tables are arranged according to complementary angles. The symbols that Regiomontanus used for degrees and minutes might be noted.

Arcus G. m̃.	Sinus
36 0	352 671 151
54 0	485 410 197
18 0	185 410 197
72 0	570 633 909

Arcus G. m̃.	Sinus
12 0	124 747 015
78 0	586 888 561
6 0	627 170 078
84 0	596 713 137

Arcus G. m̃.	Sinus
35 15	346 287 114
43 45	489 984 933
24 0	244 041 986
66 0	548 127 275

The tables of sines for 6 000 000 have a different arrangement:

G. m̃	0 Sinus	portio unius 2 10	1 Sinus	portio unius 2 10	2 Sinus	portio unius 2 10
0	0	29 1	104715	29 1	209397	29 1
1	1745		106460		211141	
2	3491		108205		212885	

The tables of sines for 10 000 000 has the same arrangement as the preceding table.

b. Tabulae directionum

The Tabulae directionum is also a posthumous work of Regiomontanus. Thirty-one astronomical problems illustrate the use of the tables which form the major part of the text. The problems are of this nature:

> To briefly find the declination of a planet whose place in the zodiac is known.
>
> To calculate in general the declination of some fixed star or planet proposed.
>
> To calculate easily the right ascension of some fixed planet.

For the purpose of this study a comparison was made of the different editions of the Tabulae directionum possessed by the University of Michigan. The introductory material in the different editions varies, but the text proper is unchanged except in the French edition. The editions investigated were: the Augsburg edition of 1490, the 1559 edition printed at Tübingen, the two Wittenberg editions of 1584 and 1606 respectively, and the Primum mobile of Erasmus Schreckenfuchs printed at Basle in 1567 which includes the Tabulae directionum of Regiomontanus.

A French edition was printed at Paris in 1626; the translation into French was made by D. Henrion. In this edition each problem is explained; the tables are not arranged as in the original work; and the word "tangent" is used instead of "tabula fecunda." The whole treatise is preceded by a brief instruction on spherical geometry and trigonometry.

Although Regiomontanus did not use the tangent function in De triangulis libri quinque, the "tabula fecunda" in the Tabulae directionum is evidence enough that he knew and appreciated this function. That he was conscious of its importance can be inferred from the last sentence of his tenth problem:

> It is not in vain that this table is called 'fruitful,' for, by resembling a tree, it is accustomed to produce great and admirable effects.

Gr.	0 Sinus	portio unius 2 10	1 Sinus	portio unius 2 10	2 Sinus	portio unius 2 10	3 Sinus	portio unius 2 10	4 Sinus	portio unius 2 10
0	0	29	104715	29	209397	29	314016	29	418540	29
1	1745		106460		211141		315759		420281	
2	3491		108205		212885		317502		422022	
3	5236		109950		214630		319244		423763	
4	6982		111695		216374		320987		425504	
5	8727		113440		218118		322730		427245	
6	10472		115185		219863		324473		428986	
7	12218		116930		221606		326216		430727	
8	13963		118675		223351		327958		432467	
9	15709		120420		225095		329701		434208	
10	17454		122165		226839		331444		435949	
11	19199		123910		228583		333187		437690	
12	20944		125655		230327		334929		439430	
13	22690		127400		232071		336672		441171	
14	24435		129145		233815		338414		442911	
15	26180		130890		235559		340157		444652	
16	27925		132635		237303		341899		446392	
17	29671		134380		239047		343642		448133	
18	31416		136124		240791		345386		449873	
19	33162		137869		242535		347127		451614	
20	34907		139614		244279		348869		453354	
21	36652		141359		246023		350611		455094	
22	38397		143104		247767		352354		456834	
23	40143		144848		249510		354098		458575	
24	41888		146593		251254		355838		460315	
25	43633		148338		252998		357580		462055	
26	45378		150083		254742		359322		463795	
27	47123		151828		256485		361064		465535	
28	48869		153572		258229		362807		467275	
29	50614		155315		259972		364549		469015	
30	52359		157062		261716		366291		470755	
31	54104		158807		263460		368033		472495	
32	55850		160551		265203		369775		474235	
33	57595		162296		266947		371517		475974	
34	59341		164040		268690		373259		477714	
35	61086		165785		270434		375001		479454	
36	62831		167530		272178		376743		481194	
37	64576		169274		273921		378485		482933	
38	66321		171019		275665		380226		484673	
39	68067		172763		277408		381968		486412	
40	69812		174508		279152		383710		488152	
41	71557		176253		280895		385452		489871	
42	73302		177997		282639		387194		491631	
43	75048		179742		284382		388935		493370	
44	76793		181486		286126		390677		495110	
45	78538		183231		287869		392419		496849	
46	80283		184975		289612		394161		498588	
47	82028		186720		291355		395902		500327	
48	83774		188464		293099		397644		502067	
49	85519		190209		294842		399385		503806	
50	87264		191953		296585		401127		505545	
51	89009		193697		298328		402868		507284	
52	90754		195442		300071		404610		509023	
53	92500		197186		301815		406351		510762	
54	94245		198931		303558		408091		512501	
55	95990		200675		305301		409834		514240	
56	97735		202419		307044		411574		515979	
57	99480		204164		308787		413316		517718	
58	101225		205908		310530		415058		519454	
59	102970		207653		312273	29	416799		521195	
60	104715		209397		314016	29	418540		522934	

Arcus. G.	m.	Sinus.
36	0	352671151
54	0	485410197
18	0	185410197
72	0	570633909
9	0	93860679
81	0	592613004
4	30	47075458
85	30	598150400
2	15	23555889
87	45	599537422
27	0	272394297
63	0	534603915
13	30	140067218
76	30	583421952
6	45	70522438
83	15	595841074
40	30	389668829
49	30	456243579
20	15	207670234
69	45	562914802
42	45	407290447
47	15	440593506
31	30	313499140
58	30	511584098
15	45	162864270
74	15	577473142
38	15	371456371
51	45	471190159
24	45	251195842
65	15	544885904
29	15	291727744
60	45	523497605

Arcus. G.	m.	Sinus.
12	0	124747015
78	0	586888561
6	0	62717078
84	0	596713137
3	0	31401574
87	0	599177721
1	30	15706169
88	30	599794394
45	0	7853773
89	15	599948596
39	0	377592235
51	0	466287577
19	30	200284116
70	30	565584895
9	45	101609702
80	15	591333635
42	0	401478364
48	0	445886895
21	0	215020770
69	0	560148256
10	30	109341315
79	30	589952945
5	15	54900971
84	45	597482957
43	30	423012745
46	30	437224623
22	45	222334482
68	15	557285732
44	15	418674276
45	45	429781166
25	30	258306658
64	30	541551171
12	45	132418461
77	15	585205392

Arcus. G.	m.	Sinus.
39	15	346287114
54	45	489984933
24	0	244041986
66	0	548127275
34	30	339843742
55	30	494475713
17	15	17792 945
72	45	573011967
39	45	383663407
50	15	461305099
23	15	236846314
66	45	551274726
32	15	320168709
57	45	507436663
33	0	326783421
57	0	703202341
16	30	179409207
73	30	575291841
8	15	86095573
81	45	593790832
27	45	279368712
62	15	530992582
28	30	286295256
61	30	527290268
14	15	147691976
75	45	581538546
36	45	358994760
53	15	480752288
30	45	306775852
59	15	515643849

Plate 6. Regiomontanus, edited by Daniel Santbech, Basle, 1561.

INSTRVMENTVM SINVVM, SEV PRÍMI MOBILIS, NV

PER A. PETRO APIANO INVENTVM; NVNC AVTEM abeodem diligenter recognitum & locupletatum. Cuius tractatione, cum genuinus & latíssime patens Sinuum usus clarissime deprehenditur, tum uero quícquíd earundem Sínuum auxílio in rebus astronomícis ratiocinando indagari potest, facíllime ac celerrime conficitur.

Adiectus est & Quadrans Vniuersalis seu Generalis, noua facie, in eundem usum fabrefactus. cuius usu, quícquíd ex tabulis Sinuum cõpartãdo elici poterat, sine negotio, sínetq; numerorũ molestiore tractatiõe facílime gíspicitur.

Norimbergæ apud Iohan. Petreium, anno M. D. XLI.

Tabula Sinuum rectorum siue semichordarum minutim exactis.

m	30 Sinus	31 Sinus	32 Sinus	33 Sinus	34 Sinus	35 Sinus	36 Sinus	37 Sinus	38 Sinus	39 Sinus
0	50000	51503	52991	54463	55919	57357	58778	60181	61566	62932
1	50015	51518	53006	54478	55934	57373	58793	60196	61581	62947
2	50030	51533	53021	54492	55948	57387	58808	60210	61595	62961
3	50044	51547	53035	54507	55963	57402	58822	60225	61610	62976
4	50059	51562	53050	54521	55977	57416	58837	60239	61624	62990
5	50073	51576	53064	54536	55992	57431	58851	60254	61639	63005
6	50088	51591	53079	54550	56006	57445	58866	60268	61653	63019
7	50102	51605	53093	54565	56021	57460	58880	60283	61668	63034
8	50117	51620	53108	54579	56035	57474	58895	60297	61682	63048
9	50131	51634	53122	54594	56050	57489	58909	60312	61697	63063
10	50146	51649	53137	54608	56064	57503	58924	60326	61711	63077
11	50160	51663	53151	54623	56079	57518	58938	60341	61726	63092
12	50175	51678	53166	54637	56093	57532	58953	60355	61740	63106
13	50189	51692	53180	54652	56108	57547	58967	60370	61755	63121
14	50204	51707	53195	54666	56122	57561	58982	60384	61769	63135
15	50218	51721	53209	54681	56137	57576	58996	60399	61784	63150
16	50233	51736	53224	54695	56151	57590	59011	60413	61798	63164
17	50247	51750	53238	54710	56166	57605	59025	60428	61813	63179
18	50262	51765	53253	54724	56180	57619	59040	60442	61827	63193
19	50276	51779	53267	54739	56195	57634	59054	60457	61842	63208
20	50291	51794	53282	54753	56209	57648	59069	60471	61856	63222
21	50305	51808	53296	54768	56224	57663	59083	60486	61871	63237
22	50320	51823	53311	54782	56238	57677	59098	60500	61885	63251
23	50334	51837	53325	54797	56253	57692	59112	60515	61900	63266
24	50349	51852	53340	54811	56267	57706	59127	60529	61914	63280
25	50363	51866	53354	54826	56282	57721	59141	60544	61929	63295
26	50378	51881	53369	54840	56296	57735	59156	60558	61943	63309
27	50392	51895	53383	54855	56311	57750	59170	60573	61958	63324
28	50407	51910	53398	54869	56325	57764	59185	60587	61972	63338
29	50421	51924	53412	54884	56340	57779	59199	60602	61987	63353
30	50436	51939	53427	54898	56354	57793	59214	60616	62001	63367
31	50450	51953	53441	54913	56369	57808	59228	60631	62016	63382
32	50465	51968	53456	54927	56383	57822	59243	60645	62030	63396
33	50479	51982	53470	54942	56398	57837	59257	60660	62045	63411
34	50494	51997	53485	54956	56412	57851	59272	60674	62059	63425
35	50508	52011	53499	54971	56427	57866	59286	60689	62074	63440
36	50523	52026	53514	54985	56441	57880	59301	60703	62088	63454
37	50537	52040	53528	55000	56456	57895	59315	60718	62103	63469
38	50552	52055	53543	55014	56470	57909	59330	60732	62117	63483
39	50566	52069	53557	55029	56485	57924	59344	60747	62132	63498
40	50581	52084	53572	55043	56499	57938	59359	60761	62146	63512
41	50595	52098	53586	55058	56514	57953	59373	60776	62161	63527
42	50610	52113	53601	55072	56528	57967	59388	60790	62175	63541
43	50624	52127	53615	55087	56543	57982	59402	60805	62190	63556
44	50639	52142	53630	55101	56557	57996	59417	60819	62204	63570
45	50653	52156	53644	55116	56572	58011	59431	60834	62219	63585
46	50668	52171	53659	55130	56586	58025	59446	60848	62233	63599
47	50682	52185	53673	55145	56601	58040	59460	60863	62248	63614
48	50697	52200	53688	55159	56615	58054	59475	60877	62262	63628
49	50711	52214	53702	55174	56630	58069	59489	60892	62277	63643
50	50726	52229	53717	55188	56644	58083	59504	60906	62291	63657
51	50740	52243	53731	55203	56659	58098	59518	60921	62306	63672
52	50755	52258	53746	55217	56673	58112	59533	60935	62320	63686
53	50769	52272	53760	55232	56688	58127	59547	60950	62335	63701
54	50784	52287	53775	55246	56702	58141	59562	60964	62349	63715
55	50798	52301	53789	55261	56717	58156	59576	60979	62364	63730
56	50813	52316	53804	55275	56731	58170	59591	60993	62378	63744
57	50827	52330	53818	55290	56746	58185	59605	61008	62393	63759
58	50842	52345	53833	55304	56760	58199	59620	61022	62407	63773
59	50856	52359	53847	55319	56775	58214	59634	61037	62422	63788
60	50871	52374	53862	55333	56789	58228	59649	61051	62436	63802

Tabula Sinuum rectorum siue semichordarum minutim exactis.

m	40 Sinus	41 Sinus	42 Sinus	43 Sinus	44 Sinus	45 Sinus	46 Sinus	47 Sinus	48 Sinus
0	64278	65605	66913	68199	69465	70711	71933	73135	74314
1	64301	65627	66935	68221	69487	70733	71955	73157	74336
2	64323	65649	66957	68243	69509	70755	71976	73177	74357
3	64345	65671	66979	68265	69531	70777	71998	73199	74379
4	64367	65693	67001	68287	69553	70799	72020	73221	74401
5	64390	65715	67023	68308	69575	70821	72042	73243	74422
6	64412	65737	67044	68330	69597	70843	72064	73264	74444
7	64434	65759	67066	68352	69619	70865	72085	73286	74465
8	64456	65781	67088	68374	69641	70887	72107	73308	74487
9	64479	65803	67110	68396	69663	70909	72129	73329	74508
10	64501	65825	67132	68418	69684	70931	72151	73351	74530
11	64523	65847	67154	68440	69706	70953	72172	73372	74551
12	64545	65868	67176	68462	69728	70975	72194	73394	74573
13	64567	65890	67197	68483	69750	70997	72216	73415	74594
14	64590	65912	67219	68497	69772	71019	72238	73437	74616
15	64612	65934	67241	68518	69794	71041	72259	73458	74637
16	64634	65956	67258	68539	69799	71063	72281	73480	74659
17	64656	65978	67279	68560	69820	71079	72302	73501	74680
18	64678	66000	67301	68581	69841	71099	72324	73491	74663
19	64701	66021	67323	68603	69862	71100	72346	73511	74683
20	64723	66043	67344	68624	69883	71120	72336	73510	74702
21	64745	66065	67365	68645	69903	71141	72358	73532	74723
22	64767	66087	67387	68666	69914	71161	72377	73570	74741
23	64789	66109	67408	68687	69945	71181	72397	7379	74760
24	64811	66131	67430	68708	69956	71202	72417	73609	74779
25	64834	66153	67451	68729	69987	71223	72437	73629	74799
26	64856	66174	67473	68751	70007	71243	72457	73649	74818
27	64878	66196	67494	68772	70028	71263	72477	73669	74837
28	64900	66218	67516	68793	70049	71284	72497	73688	74857
29	64922	66240	67537	68814	70070	71304	72517	73708	74876
30	64944	66262	67559	68835	70090	71324	72537	7372	74895
31	64966	66284	67580	68856	70111	71345	72557	73747	74914
32	64989	66305	67601	68877	70132	71365	72577	73767	74934
33	65011	66327	67623	68898	70153	71386	72597	73786	74953
34	65033	66349	67644	68919	70173	71406	72617	73806	74972
35	65055	66370	67666	68940	70194	71426	72637	73825	74991
36	65077	66392	67687	68961	70214	71447	72657	73845	75011
37	65099	66414	67709	68983	70235	71467	72677	73864	75030
38	65121	66435	67730	69004	70256	71488	72697	73884	75049
39	65143	66457	67752	69025	70277	71508	72717	73904	75068
40	65165	66479	67773	69046	70298	71528	72737	73923	75088
41	65187	66501	67794	69067	70318	71548	72757	73943	75107
42	65209	66523	67815	69088	70339	71569	72777	73963	75126
43	65231	66544	67837	69109	70360	71589	72797	73982	75145
44	65253	66566	67858	69130	70380	71609	72817	74002	75164
45	65275	66588	67880	69151	70401	71630	72837	74021	75183
46	65298	66609	67901	69172	70422	71650	72857	74041	75203
47	65320	66631	67922	69193	70443	71670	72876	74060	75222
48	65342	66653	67944	69214	70463	71691	72896	74080	75241
49	65364	66674	67965	69235	70484	71711	72916	74100	75260
50	65386	66696	67986	69256	70504	71732	72936	74119	75279
51	65408	66718	68008	69277	70525	71752	72956	74139	75299
52	65430	66739	68029	69298	70545	71772	72976	74158	75318
53	65452	66761	68050	69319	70566	71793	72996	74178	75337
54	65474	66783	68073	69340	70587	71813	73016	74197	75356
55	65496	66804	68094	69361	70607	71833	73036	74217	75375
56	65518	66826	68114	69382	70628	71853	73055	74236	75394
57	65540	66848	68135	69401	70648	71873	73075	74256	75413
58	65562	66869	68156	69423	70669	71893	73095	74275	75432
59	65583	66891	68178	69444	70690	71913	73115	74295	75451
60	65605	66913	68199	69465	70711	71933	73135	74314	75470

Plate 7. Petrus Apianus. Instrumentum sinuum, seu primi mobilis, 1541.

The following is a portion of the "Tabula Fecunda":

NUMERUS G		NUMERUS G		NUMERUS G	
0	00000	31	60086	61	180402
1	1745	32	62486	62	188075
2	3492	33	64940	63	196263
3	5240	34	67452	64	205034

The "tabula fecunda" is used in solving Problem 10: "To calculate the oblique ascension of some star which is in the horizon."

For this purpose we will make use of the difference of ascension when the star is not above the equator. If we wish to suppose the oblique ascension of a fixed star having its declination greater than 32 degrees, turn to the 'tabula fecunda.' Suppose the elevation is 48 degrees and the declination is 9 degrees 51 minutes, I find the number 111062 and by a double entry another number 17364; multiply one by the other, the product is 1928480568; this number multiplied by 6 is 11570883408. According to custom I reject the last six numbers and add one.[58] 11571 is the sine of the difference of ascension whose arc is 11 degrees 7 minutes, from which the oblique ascension can be found. It is not in vain...

tan 10° = 17633
tan 9° = 15838

tan 9° 51' = 17364

(This number is found by the usual method of interpolation.)

c. BRIEF ACCOUNT OF OTHER WORKS

Regiomontanus produced a considerable number of works. George Tanstetter, doctor and astronomer, who lived at the beginning of the sixteenth century, listed them in 1514 at the end of a eulogy on Regiomontanus inserted in the tables of eclipses of Peurbach; the list comprises twenty-one translations and twenty-five original works.[59] From the latter group, the following might be cited:

Ephemerides astronomicae ab anno 1475 ad annum 1506. Nuremberg, 1474.

Kalendarium novum, quo promuntur conjunctiones verae atque oppositiones luminarium itemque eclipses scrumdem figuratae... Nuremberg, 1476.[60]

Problemata XVI de cometae longitudine, magnetudine et loco vero. Ed. Schonerus, Nuremberg, 1531. The appearance of the comet of 1531, later known as Halley's comet, led to the publication of this work written in 1476.

Epitome in Almagestum Ptolemaei, Venice, 1496.

d. CONCLUSION

Science is indebted to Regiomontanus for the keen interest he took in it, his prolific work, and his painstaking efforts to advance it by publications of important works of many writers. Regiomontanus always stayed within the bounds of the geocentric system.[61] His contributions to astronomy justify calling him the greatest of the

58. It was such expressions as this that lead historians of mathematics to assert that Regiomontanus invented decimal fractions.
59. F. Boquet, op. cit., p. 239.
60. The editions at University of Michigan are: Venice, E. Ratdolt, 1483 and 1485.
61. See Johann Schöner, Opusculum geographicum ex diversorum libris et cartis collectum (Nuremberg, 1533) 20 ff 4°. Compare with J. L. E. Dreyer, History of the Planetary Systems from Thales to Kepler (Cambridge: University Press), pp. 290-91.

predecessors of Copernicus. His publication, the De triangulis libri quinque, is his greatest contribution to science. This work severed permanently the traditional bonds that kept trigonometry the servant of astronomy.

4. JOHANN WERNER

The center of scientific and cultural life of Germany at the opening of the sixteenth century was at Nuremberg in the home of Willibald Pirkheimer,[62] whose role in effecting the publication of the works of Regiomontanus was not a minor one.[63] Johann Werner, a student of geography and mathematics, belonged to the Pirkheimer circle.[64] From Nuremberg, where he was born in 1468, he went to Rome, mindful of the fact that here Regiomontanus had acquired great learning.[65] After five years of study, he returned to his own country to take over an ecclesiastical charge; serving in the capacity of pastor of St. John's, he did not neglect his studies.[66]

Werner wrote a spherical trigonometry in five books under the title, De triangulis maximorum circulorum constructes libri V, but unfortunately he could find no one who would print his treatise.[67] After Werner's death his manuscript was in the possession of George Hartmann,[68] who gave it to George Joachim Rheticus, professor of mathematics in Wittenberg, when he was in Nuremberg in May 1542.[69] The influence of Rheticus on the development of trigonometry will be discussed in the next chapter; how much he might have borrowed from Werner will also be considered.

None of Werner's other works are of a purely mathematical nature; he made several translations, of which the first book of the geography of Ptolemy is one. In 1522 he published the De motu octavae spherae tractatus primus;[70] he was interested in the reform of the calendar; and he promoted a method of measuring lunar distances by longitudes which proved helpful to navigators.[71]

5. PETER APIAN (BIENEWITZ)

Peter Apian, the youngest of the contemporaries of Copernicus, was born at Leisnig in Saxony, 1475.[72] He became professor of science in Ingolstadt and stood high in the estimation of Charles V.[73] Apian's knowledge of trigonometry can be gleaned from the definitions he sets forth in his Instrumentum sinuum seu primi mobile, printed by Johann Petreus in 1541. His definitions are:

> The diameter of a circle is the straight line passing through the center of a circle and dividing it into two equal parts.
>
> An arc is any portion of the periphery of a circle.
>
> A chord is a straight line cutting a circle into two unequal parts.
>
> The 'sinus' is a straight line measured according to the length or value of the diameter.

62. Willibald Pirkheimer (1470-1530) was a celebrated German humanist; he studied jurisprudence at Padua; his influence in Germany is likened to that of Erasmus and Reuchlin (Nouvelle Biographie Générale [Paris: F. Didot, 1857], vol. XC, pp. 303-307).
63. A. von Braunmühl, op. cit., p. 133.
64. Idem.
65. F. Boquet, op. cit., p. 244.
66. Idem.
67. A. von Braunmühl, loc. cit.
68. George Hartmann (1480-1543), vicar of St. Sebaldus Church in Nuremberg is considered the discoverer of the inclination of the magnetic needle (A. von Braunmühl, loc. cit.).
69. Idem.
70. Copernicus criticized this work in his Letter against Werner; see p. 42.
71. F. Boquet, loc. cit.
72. Siegmund Günther, Peter und Philipp Apain (Praque: Gregr, 1882), p. 5.
73. Op. cit., p. 12.

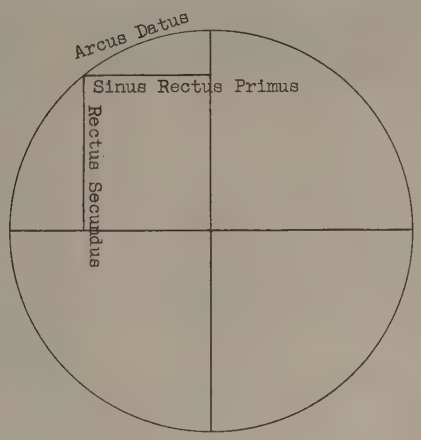

Fig. 43

The 'sinus totus' or 'perfectus' is the semidiameter of a circle divided into several parts.

The 'sinus' is divided into 'sinus rectus' and 'sinus versus.'

The 'sinus rectus' is subdivided into the 'sinus primus' and 'sinus secundus.'

The 'sinus rectus primus,' drawn from the middle of the double chord to the arc is the 'sinus' or half chord with respect to the whole arc.

The 'sinus versus' is the portion of the diameter included between the 'sinus rectus' and periphery, namely, the 'sagitta' of the arc.

The 'sinus rectus secundus' is the remainder of any arc after it is subtracted from 90, or the part which is complementary to the fourth of a circle which is always equal to that part of the diameter from the center of a circle to the 'sinus rectus primus.'

This treatise of Apian contains a table of sines covering nine pages for a radius equal to 100 000 and an interval of one minute for angles of the first quadrant.

Von Braunmühl[74] asserts that Apian's knowledge of trigonometry indicates no advance over that of Regiomontanus; moreover, Apian was not familiar with the advantages of the "tabula fecunda."

The Astronomium Caesareum[75] of Apian might also be mentioned, even though the absence of mathematics is a striking feature of the work. Apian's purpose in writing this book was to make astronomy easy for all, and more particularly for his patrons, Emperor Charles V of Rome and King Ferdinand of Spain, to whom the book was dedicated and for whom it was named. It was printed at Ingoldstadt, May, 1540.

Apian's reason for keeping this book free from mathematics is found in his preface:

> We see how much mathematics are hated because of their seeming difficulty. We tried to simplify and planned by what new methods help could be found for those whom arithmetic baffles.

74. Op. cit., p. 139.

75. This work is described at some length by Van Ortoy in "Bibliographie de l'oeuvre de Pierre Apian," Bibliograph Moderne, 5:327-329, July-October, 1901.

See also S. A. Ionides, "Caesar's Astronomy (Astronomium Caesareum) by Peter Apian, 1540, Ingolstadt," Osiris, 1-356-389.

In this work Apian records his most noted astronomical discovery, that comets always point their tails away from the sun. He also uses the meteoroscope, an instrument invented by him for the solution of spherical triangles.

Thus the history of the early sixteenth century depicts Germany as one of the leading countries sponsoring the advance of science, still only a few years after the death of Regiomontanus

> ...a quiet student at the shore of the Baltic, on the very outskirts of civilization, was preparing to kindle the light which was to illuminate the unverse and show to astonished humanity the earth moving through space.[76]

The task of furthering the work of Regiomontanus fell to Copernicus; the perfecting of it passed to Rheticus, the loyal friend and co-laborer of Copernicus.

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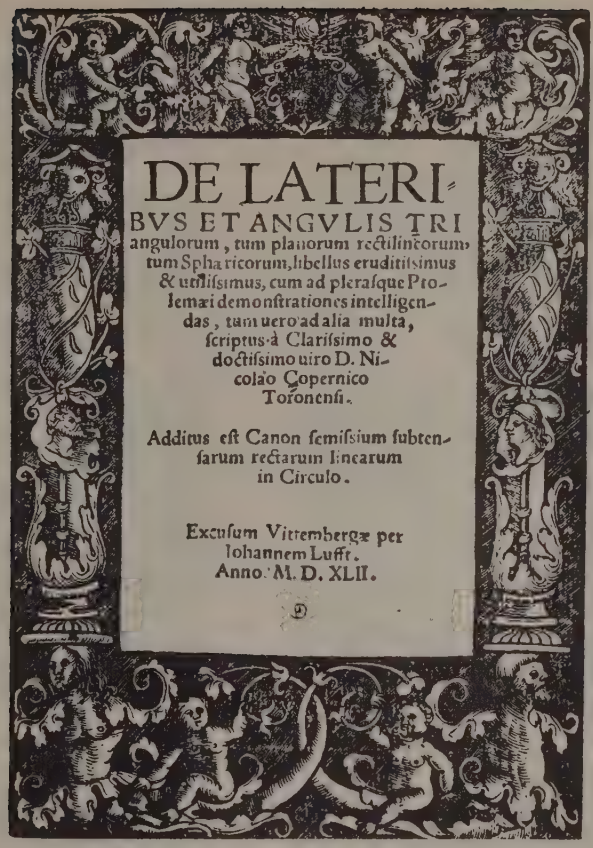
76. J. L. E. Dreyer, op. cit., p. 304.



DE LATERI-
BVS ET ANGVLIS TRI
angulorum, tum planorum rectilinẽorum, tum Sphaericorum, libellus eruditiſsimus & utiliſsimus, cum ad pleraſque Proiemaei demonſtrationes intelligendas, tum uero ad alia multa, ſcriptus à Clariſsimo & doctiſsimo uiro D. Nicolao Copernico Torunenſi.

Additus eſt Canon ſemiſsium ſubtenſarum rectarum linearum in Circulo.

Excuſum Vittembergæ per Iohannem Lufft. Anno. M. D. XLII.

CANON SVBTENSARVM IN CIRCVLO RECTARVM LINEARVM.

	0	1	2	3	4	
1	250	177413	353607	529816	705967	59
2	5015	180141	355429	531570	707630	58
3	8747	182850	357178	533207	709170	57
4	31636	185532	360523	534980	709172	55
5	34544	188186	363530	537184	712078	55
6	37451	190811	366417	540289	714978	54
7	40362	194083	369144	541693	717876	53
8	43271	197792	372159	546558	720777	52
9	46180	200700	375156	549503	723673	51
10	49088	203608	378064	552407	726579	50
11	51997	206517	380971	555312	729480	49
12	54906	209425	383878	558216	732381	48
13	57815	212333	386795	561120	735282	47
14	60724	215241	389692	564024	738183	46
15	63632	218149	392598	566928	741084	45
16	66541	221057	395505	569832	743985	44
17	69450	223965	398412	572736	746886	43
18	72359	226873	401318	575640	749787	42
19	75268	229781	404225	578544	752688	41
20	78177	232685	407131	581448	755588	40
21	81086	235597	410038	584352	758489	39
22	83995	238504	412944	587256	761389	38
23	86904	241412	415851	590160	764290	37
24	89813	244320	418757	593064	767190	36
25	92721	247228	421663	595967	770090	35
26	95630	250136	424570	598871	772991	34
27	98539	253043	427476	601775	775891	33
28	101448	255951	430382	604678	778791	32
29	104357	258860	433288	607582	781691	31
30	107265	261768	436194	610485	784591	30
	89	88	87	86	85	

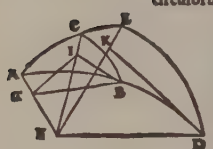
	0	1	2	3	4	
31	90174	264677	439100	613188	787498	29
32	93083	267585	442006	616192	790398	28
33	95992	270493	444912	619196	793298	27
34	98901	273401	447818	622099	796198	26
35	101809	276308	450724	625002	799098	25
36	104718	279216	453630	627905	801998	24
37	107627	282124	456536	630808	804889	23
38	110536	285032	459442	633711	807789	22
39	113445	287940	462348	636614	810685	21
40	116353	290847	465253	639517	813587	20
41	119262	293755	468159	642420	816486	19
42	122171	296663	471065	645323	819385	18
43	125079	299570	473970	648226	822284	17
44	127988	302478	476876	651129	825183	16
45	130896	305385	479781	654031	828082	15
46	133805	308293	482687	656934	830981	14
47	136714	311200	485592	659837	833880	13
48	139622	314108	488498	662739	836778	12
49	142531	317015	491403	665642	839677	11
50	145439	319922	494308	668544	842575	10
51	148348	322830	497214	671447	845474	9
52	151257	325737	500119	674349	848372	8
53	154165	328645	503024	677251	851271	7
54	157074	331552	505929	680153	854169	6
55	159982	334459	508834	683056	857067	5
56	162891	337367	511740	685957	859965	4
57	165799	340274	514645	688859	862863	3
58	168708	343181	517550	691761	865761	2
59	171616	346088	520455	694663	868659	1
60	174525	348995	523360	697565	871557	0
	89	88	87	86	85	

D

Plate 8. Copernicus, De Lateribus et Angulis Triangulorum, Wittemberg. 1542. With table of sines. Edited by Georg Joachim Rheticus. First part of the Copernicus De revolutionibus....

angulo, ad ſubtenſam duplo alterius rectum angulum cõprehendentium, eſt, ſicut dimetiens Sphæræ ad eam, quæ duplum anguli ſub reliquo & primo lateribus comprehen ſi in maximo Sphæræ circulo ſubtendit.

Eſto nanque triangulum Sphęricum a b c, cuius c angulus rectus exiſtat. Dico quod ſubtenſa dupli a b ad ſubtenſam dupli b c eſt ſicut dimetiens Sphæræ, ad eam quæ in maximo circulo duplum anguli b a c ſubtendit. Facto in a polo, deſcribatur circumferentia maximi circuli d e, & compleantur quadrantes circulorum a b d & a c e. Et ex centro Sphæræ f agantur comunes circulorum ſectiones f a ipſorum a b d & a c e, ipſorum autem a c e & d e ſit f e, atque f d ipſorum a b d & d e. Inſuper & f c circulorum a c & b c. Deinde ad angulos rectos agantur b g ipſi f a, b i ipſi f c, & d k ipſi f e, & connectatur g i.

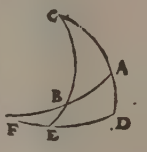


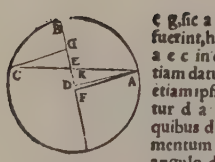
Quoniam igitur ſi circulus circulum per polos ſecat, ad angulos rectos ipſum ſecat, erit angulus qui ſub a e d comprehenditur rectus, & a c b per hypotheſim, & utrumq; planum e d f, & b c f rectum ad ipſum a e f. Quapropter ſi ex ſigno ipſi f k e communi ſegmento ad rectos angulos in ſubiecto plano recta linea excitaretur, comprehendet quoq; cum k d angulum rectum, per rectorum ad inuicem planorum definitionem. Quapropter etiam ipſa k d per iiij. vndecimi Euclidis ad a e f recta eſt. Ac eadem ratione b i ad idem planum erigitur, & idcirco ad inuicem ſunt d k & b i per vi. eiuſdem. Verum etiam g b, ad f d, eo q; f g b, & g f d anguli ſunt recti, erit per x. undecimi Euclidis, angulus f d K ipſi g b i æqualis. At qui ſub f k d rectus eſt, & g i b per definitionem erectæ lineæ. Similium igitur triangulorum proportionalia ſunt latera, & ut d f ad b g, ſic d k ad b i. Et b i eſt dimidia ſubtendentis duplum c b circumferentiam, quoniam ad angulum rectum eſt, ad eam, quæ ex centro f, & eadem ratione b g dimidia ſubtendentis duplum latus b a, & d k ſemiſsis ſubtendentis duplam d e, ſiue angulum dupli a, atque d f dimidia diametri ſphæræ. Patet igitur quod ſubtenſa dupli ipſius a b, ad ſubtenſam dupli b c, eſt ſicut dimetiens ad eam quæ duplum anguli a ſiue interceptę circumferentię d e ſubtendit, quod demonſtraſſe fuerit opportunum.

IIII.

In quocunque triangulo rectum angulum habente, alius inſuper angulus fuerit datus, cum quolibet latere, reliquus etiam angulus cum reliquis lateribus dabitur. Sit enim triangulum a b c habens angulum a rectum, & cum ipſo etiam alterutrum utputa b datum. De latere uero dato trifariam ponimus diuiſionem, aut enim fuerit, qui datis adiacet angulis, ut a b, aut recto tantum, ut a c, aut qui opponitur recto, ut b c. Sit ergo primum a b latus datum, & facto in c polo deſcribatur circumferentia maximi circuli d e, & completis quadrantibus c a d & c b e, producantur a b & d e donec ſe inuicem ſecent in f ſigno. Erit ergo uiciſsim in f polus ipſius c a d, eo q; circa a & d ſunt anguli recti. Et quoniam ſi in ſphæra maximi orbes ad rectos ſeſe inuicem ſecuerint angulos, bifariam & per polos ſe iuuicem ſecant. Sunt ergo & a b f & d e f quadrantes circulorum, cum q; data ſit a b, datur & reliqua quadrantis b f, & angulus e b f ad uertícem ipſi a b c dato æqualis. Sed per præcedentem demonſtrationem ſubtenſa dupli b f ad ſubtendentem dupli e f, eſt ſicut dimetiens ſphæræ ad ſubtendentem duplum anguli e b f. Sed tres earum datæ ſunt, dimetiens ſphæræ, duplę b f, atq; anguli dupli e b f, ſiue ſemiſſes ipſorum. Datur ergo per xvi. ſexti Euclidis etiam dimidia ſubtendentis duplam e f per canonem ipſa e f circumferentia, & reliqua quadrantis d e, ſiue angulus c quæſitus. Eodem modo ac uiciſsim ſunt ſubtenſæ duplicium d e ad a b, & e b c ad c b. Sed tres iam datæ ſunt d e, a b, & e b c quadrantes circuli, datur ergo & quarta ſubtendens duplum c b, & ipſum latus c b quæſitum. Et quoniam ſubtenſæ duplicium ſunt ipſo-

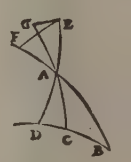
rum





e g, ſic a e ad e c. Quibus igitur numeris a f vel g c data fuerint, habebimus in iſdem a e & e c, dabitur ex his tota a e c in eiſdem. Sed ipſa ſubtendens a b c circumferentiam datur in partibus, quibus quę ex centro d e b, quibus etiam ipſius a c dimidia a k, & reliqua e k. Coniungantur d a & d k, quæ etiam dabuntur in eiſdem partibus, quibus d b, tanquam ſemiſsis ſubtendentis reliquum ſegmentum ipſius a b c à ſemicirculo, comprehenſum ſub angulo d a k & angulus igitur a d k datur comprehendens, dimidiam a b c circumferentiam. Sed & trianguli duobus lateribus datis & angulo e k d recto, dabitur etiam e d k, hinc totus ſub e d a angulus comprehendens a b circumferentiam, qua etiam reliqua c b conſtabit, quorum expetebatur demonſtratio.

nem habebit, quam ſemiſsis ſub duplo angulo e a f ad ſemiſſem ſub duplo anguli e a g. Et quoniam & f e, e g circumferentiæ datę ſunt, ſunt enim reſidua, quibus anguli a & b differunt à rectis. Habebimus ergo ex his rationem angulorum e a f & e a g, hoc eſt, b a d ad c a d, qui illis ad verticem ſunt, datos. Totus autem b a c datus eſt. Per præcedens igitur Theorema etiam b a d & c a d anguli dabuntur.

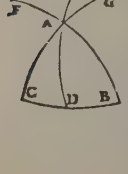


Deinde per quintum, latera a b, b c, a c, c d, totuḿq; b c aſſequemur.

FINIS.

XV.

Trianguli datis omnibus angulis, etiam nullo recto, dantur omnia latera. Eſto triangulum a b c, cuius omnes anguli ſint dati, nullus autem eorum rectus. Aio omnia quoque latera eius dari. Ab aliquo enim angulorum ut a deſcendat per polos ipſius b c circumferentia a d, quæ ſecabit ipſum b c ad angulos rectos, ipſáq; a d cadet in triangulum, niſi alter angulorum b uel c ad baſim obtuſus eſſet, & alter acutus, quod ſi accideret, ab ipſo obtuſo deducendus eſſet ad baſim. Completis igitur quadrantibus b a f, c a g, d a e, factiſq; polis in b c, deſcribantur circumferentiæ e f, e g. Erunt igitur & circa f g anguli recti. Triangulorum igitur rectum angulum habentium erit ratio dimidiæ quæ ſub duplo a e, ad dimidiam ſub duplo e f, quæ dimidia diametri ſphæræ ad dimidiam ſubtendentis duplum anguli e a f. Similiter in triangulo a e g angulum rectum habente g, ſemiſsis quæ ſub duplo a e ad ſemiſſem, quæ ſub duplo e g, eandem habebit rationẽ, quam dimidia diametri ſphæræ ad dimidiam, quæ duplum anguli e a g ſubtendit. Per æquam igitur rationem dimidia ſub duplo e f ad dimidiam ſub duplo e g rationem



CANONSVBTENSARVM IN CIRCVLO RE-ctarum linearum.

Plate 9. Copernicus, De Lateribus et Angulis Triangulorum, Wittemberg, 1542. With table of sines. Edited by Georg Joachim Rheticus. First part of the Copernicus De revolutionibus....

Chapter III

THE TRIGONOMETRY OF COPERNICUS AND RHETICUS

1. HISTORICAL NOTE - THE GOLDEN AGE OF POLAND

The true grandeur of Poland in the first half of the sixteenth century, which is known as her "Golden Age" was due rather to the rapid development of Polish culture than to political or military success.[1] After the foundation of Cracow University - one of the glories of the country, the Alma Mater of a brilliant group of scientists, philosophers, and men of letters[2] - there arose in Poland a school of philosophy and science and later of humane literature.[3] This classical period, which had its full flower in Copernicus, differed from that of Western Europe in that it was of longer duration.[4]

> Scholars have been mystified to see Copernicus emerge from an apparent scientific obscurity of time and place. In part this results from the failure to recognize that the milieu from which Copernicus emerged was one of the most advanced intellectually in Europe. Poland had from the earliest days of its history an enthusiastic interest in education. In addition Poland was close to German and Austrian centers of astronomical studies, and also intimately connected with Italy by ecclesiastical ties. In consequence the European activity with the printing press soon found continuation in Poland.[5]

A printing press existed in Cracow in 1474, but the first permanent press was established there in 1500 by Jan Haller; this press issued mostly Latin books.[6] The first book in Polish was not printed until 1521;[7] it was Sejm zwierzat by Jan Antonin, printed in Cracow probably by Ungler.[8] After a careful inspection of lists of astronomical treatises published during the period from the invention of printing to 1543, Professor Karpinski[9] concludes that those of Polish authors and Polish institutions of learning are equalled in number only by Italian and possibly German publications.

From Wittenberg there came to Poland in 1539 a young professor of mathematics - Rheticus, whose name will always be mentioned with that of his teacher, Copernicus. In 1540 Rheticus wrote to Schöner; a section of this letter[10] was entitled "In Praise of Prussia"; here Rheticus mentioned the centers of learning[11] of sixteenth century Prussia:

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1. W. Th. Dobrzynski, Poland, Lights and Shadows in the Life of an Ancient Nation (Dublin: The Talbot Press, 1941), p. 25.
2. Ibid., p. 87.
3. A. E. Tennant, Studies in Polish Life and History (London: G. Allen and Unwin, 1924), p. 214.
4. Idem.
5. L. C. Karpinski, "The Progress of the Copernican Theory," Scripta Mathematica, vol. IX, September, 1943, pp. 139-154.
6. E. C. Corsi, Poland, Land of the White Eagle (New York: Wyndham Press, 1933), p. 122.
7. N. O. Winter, Poland of Today and Yesterday (Boston: L. C. Page and Company, 1913), p. 203.
8. See K. Estreicher, Bibliografia Polska (Cracow, 1882), vol. VIII, p. 15.
9. Op. cit.
10. Narratio prima; see p. 43.
11. "Poland at the end of the fifteenth century was a fairly important state. Not only did it wrest from the Teutonic Knights Pomerelia, or West Prussia, but East Prussia was its fief, and the grand master of the Teutonic Knights, a vassal of the Polish kings. Lithuania was little more than a dependency of Poland, and a large section of the present Ukraine was Polish territory" (Boak, Hyma, Slosson, The Growth of Western Civilization [New York: F. S. Crofts and Company, 1938], part 1, p. 415).

...Konigsberg, seat of the illustrious prince, Albert, Duke of Prussia, margrave of Bradenburg, etc., patron of all the learned and renowned men of our time; Thorn, once quite famous for its market, but now for its foster-son, my teacher; Danzig, metropolis of Prussia, eminent for the wisdom and dignity of its Council, for the wealth and splendor of its renascent literature; Frauenburg, residence of a large body of learned and pious men, famous for its eloquent and wise Bishop, the Most Reverend John Dantiscus; Marienburg, treasury of his Serene Majesty, the King of Poland; Elbing, ancient settlement in Prussia, where, too, the sacred pursuit of literature is undertaken; Kulm, famous for its literature, where the Law of Kulm had its origin.[12]

The Golden Age of Poland was a splendid epoch, distinguished in a special way by the life and work of its supreme figure, a man who belongs not only to the Polish but to all mankind.

2. NICOLAUS COPERNICUS - BRIEF BIOGRAPHICAL SKETCH

Nicolaus Copernicus was born at Thorn on February 12, 1473; his ancestors were Slavonic, and by birth Nicolaus was a citizen of Poland.[13] His father was a prosperous merchant and municipal official of Thorn;[14] his mother, Barbara Watzelrode, was a sister to Lucas Watzelrode, who became Bishop of Ermland in 1489.[15] Copernicus' early education was received at St. John's School in Thorn, but was supplemented at home by the study of the Latin and Greek languages.[16] His father died when he was ten years old, so the task of guiding and planning his career was entrusted to his maternal uncle, Bishop Watzelrode.[17]

In the autumn of 1491, Nicolaus matriculated at the University of Cracow which was famous for its high standard of science, especially mathematics and astronomy; an old academic tradition has it that here Albert Brudzewo[18] was his principal teacher.[19]

The annals of the University of Bologna show that Copernicus registered there as a student in 1496.[20] Besides attending classes in Canon Law, he continued his astronomical studies under Dominicus Maria Novara.[21] Probably Novara's love for astronomy induced the young Copernicus to decide his vocation in favor of that science.[22]

Through the instrumentality of Dominic Maria, Copernicus was named professor at the University of Rome in 1499; here his eloquence attracted a large audience who likened him to the late German scholar, Regiomontanus.[23]

In 1501 Copernicus went north to Ermland, but returned to Italy in a few months to study medicine at the University of Padua; he remained there for two years except for a few weeks he spent in Ferrara to take the doctor's degree in Canon Law.[24]

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12. Edward Rosen, Three Copernican Treatises (New York: Columbia University Press, 1939), p. 190.
13. Jean Czynski, Kopernik et ses Travaux (Paris: Libraire de Jules Renourd et cie., 1847), p. 27.
14. Edward Rosen, op. cit., p. 4.
15. Jean Czynski, op. cit., p. 29.
16. Idem.
17. Idem.
18. Albert Brudzewo (d. 1495) was the author of a commentary on Theoricae novae planetarum of Peurbach, written in 1482, printed in Venice in 1495, and reedited in Cracow in 1900 by Birkenmajer.
19. Stephen Mizwa, Great Men and Women of Poland (New York: Macmillan Co., 1942), p. 38.
20. Leopold Prowe, Nicolaus Coppernicus (Berlin: Weidmannische Buchhandlung, 1883), vol. I, part 1, p. 230.
21. Dominic Maria di Novara was born in France in 1454 or 1464 and died in Bologna fifty years later (F. Boquet, op. cit., p. 243).
22. Idem.
23. Jean Czynski, op. cit., p. 33.
24. Stephen Mizwa, op. cit., p. 43.

In 1510 Copernicus was named Canon of Frauenburg,[25] a little village situated in Polish Prussia.[26] In this position his time was divided among the occupations of canon, doctor, diplomat, and economist.

As canon, he gave proof of great piety in the performance of divine service; as doctor, he administered to the needs of his clerical friends and the poor.[27]

The Teutonic Order, under Albert of Hohenzollern, fomented disorders in the Polish provinces, especially in Ermland; Copernicus, commissioned by the Cathedral Chapter, wrote a complaint to the king on July 22, 1516 - this was the first but not the last protest of the astronomer against the raids of the Teutonic Knights.[28] Copernicus was chosen by his colleagues to be administrator of all church property, so he settled at Olsztynek, a dangerous and exposed fortress of Ermland, where he spent three years.[29]

In connection with the currency reform of Ermland and neighboring provinces, he wrote a special treatise formulating an economic law which stated that the worst form of currency in circulation regulated the value of the whole and drove all good currency out of circulation.[30]

Always the scientist despite his multifarious duties, Copernicus kept up his observations, tabulated his results, organized his ideas, and with much deliberation wrote down his theories. His friends knew the worthwhileness of the results of his hours of study and urged him to publish his works. All their endeavors were to no avail until a young Wittenberg mathematician, desiring to learn the new theories that were slowly making their way into Central Europe, journeyed to Poland and won the confidence and sympathy of the aged astronomer, who released his manuscripts and permitted a summary of their contents to be made in order to prepare the way for the acceptance of the revolutionary ideas in the work itself. This young disciple of Copernicus was George Joachim Rheticus.

3. GEORGE JOACHIM RHETICUS - BRIEF BIOGRAPHICAL SKETCH

George Joachim von Lauchen, universally known as George Joachim Rheticus, was born February 16, 1514, in the Roman province of Rhaetia.[31] At an early age he accompanied his parents to Italy; later he went to study at Zürich; in 1532 he entered the University of Wittenberg, obtained his degree, but continued his studies at Nuremberg and Tübingen until 1536 when he received the appointment of professor of mathematics at Wittenberg.[32]

In 1539 Rheticus went to Poland. He worked with Copernicus for "three years more or less."[33] He acknowledged the superior mind of Copernicus in his letter to Schöner:[34]

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25. "There is no document to show that Copernicus ever received higher orders. His medical practice, which was only private, would not speak against him being a priest; and the fact that in 1537 King Sigismund of Poland put his name on the list of four candidates for the vacant episcopal seat of Ermland makes it probable that, at least in later life, he had entered the priesthood (Catholic Encyclopedia [New York: Robert Appleton Co., 1907], vol. IV, p. 353).
26. Jean Czynski, op. cit., p. 35.
27. Idem.
28. Stephen Mizwa, op. cit., p. 41.
29. Idem.
30. Ibid., p. 42.
31. Edward Rosen, loc. cit.
32. Idem.
33. "Cum vero in Prussia, plus minus triennio egissem" (Preface which Rheticus wrote for John Werner's De triangulis sphaericis and De meteorscopiis; the preface was printed separately at Cracow in 1551, the two works by Werner were first published in Abhandlungen zur Geschichte der mathematischen Wissenschaften, 24, 1909-1910).
34. Narratio prima; see p. 43.

> ...this man whose work I am now treating is in every field of knowledge and in mastery of astronomy not inferior to Regiomontanus. I rather compare him with Ptolemy, not because I consider Regiomontanus inferior to Ptolemy, but because my teacher shares with Ptolemy the good fortune of completing, with the aid of divine kindness, the reconstruction of astronomy which he began, while Regiomontanus - alas, cruel fate - departed this life before he had time to erect his columns.

On leaving Prussia in 1541, Rheticus returned to Wittenberg to resume his teaching and supervise the separate printing of the trigonometrical portion of Copernicus' De revolutionibus orbium coelestium.[35] From Wittenberg he went to Nuremberg to set up the first sections of this great work. After his departure for Leipzig, the task of supervising the printing was undertaken by Andreas Osiander.[36] The great work was completed in 1543; Copernicus received it on May 24 just a few hours before he died.

4. SCIENTIFIC WORKS OF COPERNICUS

a. Commentariolus

The date and the composition of the Commentariolus cannot be determined; it expounds a heliocentric system independent of that in Copernicus' later work.[37] It was never printed during the author's life time, but a copy found in Vienna was published by Maximilian Curtze in 1878; Arvid Lindhagen published a Stockholm copy in 1881; on these two texts Leopold Prowe based the text from which Edward Rosen made his translation.[38]

b. Letter against Werner

The Letter against Werner and the Commentariolus are the minor astronomical works of Copernicus. The former is a critical essay on Werner's De motu octavae sphaerae tractatus primus. Manuscript copies of this letter circulated for a time; the first edition was made from a Berlin copy.[39] Jan Baranowski, 1854, and Ignacy Polkowski, 1873, made use of this edition for their texts; Maximilian Curtze edited a Vienna manuscript and from this edition Edward Rosen made his translation.[40]

c. De lateribus et angulis triangulorum

The trigonometric portion of the great astronomical work of Copernicus was printed under the title De lateribus et angulis triangulorum in 1542 at Wittenberg by Johann Lufft; Rheticus supervised the publication of this book.

d. De revolutionibus orbium coelestium

The year 1543 witnessed the appearance of De revolutionibus orbium coelestium. Its various editions are: the first edition of 1543 edited in Nuremberg by Rheticus and Osiander, the 1566 edition printed at Basle, the 1617 edition published at Amsterdam, the Warsaw edition of 1854, and the 1873 edition published at Thorn by the Academy of Sciences at Thorn. Translations, whole or in part, have been made by Karl Ludwig Menzzer in 1879 at Thorn under the title Über die Kreisbewegungen der Weltkörper; by

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35. Edward Rosen, op. cit., p. 5.
36. Osiander was the author of the anonymous Praefatiuncula which appeared in the 1543 edition; it was sixty years before its author was known; not until eighty years ago was it realized that Osiander had disagreed with Copernicus regarding the nature of astronomical hypothesis and replaced the Introduction of Copernicus by the Praefatiuncula, from which he withheld his signature (Edward Rosen, "The Ramus-Rheticus Correspondence," Journal of the History of Ideas, 1:366-67, June, 1940).
    See Gassendi, op. cit., p. 451; also, Prowe, op. cit., pp. 520-539.
37. Edward Rosen, Three Copernican Treatises (New York: Columbia University Press, 1939), p. 6.
38. Ibid., p. 7.
39. Ibid., p. 9.
40. Idem.

Alexandre Koyre at Paris in 1934, with the title Des révolutions des orbes célestes and by Charles Glenn Wallis of Annapolis, Maryland, in 1939, with the title On the Revolutions of the Celestial Spheres.[41]

5. WORKS OF RHETICUS

a. Narratic prima

The Narratio prima was a letter addressed to John Schöner, to whom Rheticus had promised some message regarding his sojourn in Poland. It was to ascertain, as the author expressed it, "whether actuality answered to report and to my own expectation." To Schöner Rheticus wrote:

> ...I shall set forth, as briefly and clearly as I can, the opinions of my teacher on the topics which I have studied.

The letter was printed in Danzig in 1540 and at Basle in 1541; it was included in the second, fourth, and fifth editions of De revolutionibus orbium coelestium - the fourth edition also has a translation into Polish - and in the 1596 and 1621 editions of Kepler's Mysterium Cosmographicum printed at Tübingen and Frankfort respectively; Prowe printed it for the eighth time in 1883 in his Nicolaus Coppernicus and from this text Edward Rosen made his translation.[42] The translation of Edward Rosen was used almost exclusively in the present work.

b. Opus palatinum de triangulis

In 1551 Rheticus resigned his position at the University of Leipzig and devoted himself principally to the calculation of trigonometric tables.[43] He calculated a table of sines, tangents, secants, and their complementary functions with the radius equal to 10000000 from 10" to 10".[44] He began the construction of a table of tangents and secants with the radius equal to 1000000000000000 for the same interval, but died before finishing them.[45] Valentine Otho (1550?-1605), a faithful pupil of Rheticus, completed the tables in 1596; they were printed as the Opus palatinum de triangulis, begun by George Joachim Rheticus and completed by L. Valentine Otho. The Thesaurus mathematicus, a table of sines, was also begun by Rheticus and completed by Otho in 1598; it was corrected and republished in 1613 by Pitiscus.[46] Dr. W. Jordan published a German edition of the Opus palatinum de triangulis in 1913.

c. Lost Works

During his stay in Prussia, Rheticus prepared a map of the region; the map is not extant, but the essay on the methods of drawing maps was published by Franz Hipler.[47]

A biography of Copernicus and an essay proving that the new astronomy did not contradict Scripture have disappeared; the loss of the former work is unfortunate, for one who had such intimate connections with the astronomer as Rheticus could have written

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41. In this work all the translations have been consulted and used. The English translation by Wallis was borrowed from St. John's College Library, Annapolis, Maryland.
42. Edward Rosen, op. cit., pp. 109-196.
43. Ibid., p. 5.
44. This table appeared in 1551 under the title Canon doctrinae triangulorum; it was printed at Leipzig by Wolfgang Gunter. In 1579 Vieta extended these tables to seven places for all six functions for every second arc; these tables appeared under the name Canon mathematicus seu ad triangula cum Appendicibus. For a description and comparison of these tables, see: Karl Hunrath, "Des Rheticus Canon doctrinae triangulorum und Vieta's Canon mathematicus," Abhandlungen zur Geschichte der Mathematischen Wissenschaften, 9:213-240, 1899.
45. Florian Cajori, op. cit., p. 132.
46. See p. 99.
47. "Chorographie des Joachim Rhaeticus," Zeitschrift für Mathematik und Physik, 21:126, 1876.

an account that would have thrown light on many of the obscurities in the life of Copernicus.

6. TRIGONOMETRY OF COPERNICUS

Before the trigonometric methods of Copernicus are considered, a comparison of De lateribus et angulis triangulorum and the trigonometric portion of De revolutionibus orbium coelestium will be given. The trigonometric work will be designated by R; the astronomical treatise by N.

R is twenty by fifteen and a half centimeters, N is twenty-six centimeters, but words per page are approximately the same. Figures in R are placed in the margin; in N text is indented on the outside of the page for the insertion of figures.

There is no consistency in the differences in abbreviations and spelling - what occurs some place in R will occur elsewhere in N, and vice versa. A few examples are:

R			N		
	Page	Line		Page	Line
inq̃	A_{iva}	6	inquam	19_b	5
cum	A_{iva}	17	cũ	19_b	16
inaequalia	A_{iva}	18	inęqualia	19_b	17
q3	A_{iva}	33	quod	19_b	33
qm̃	A_{ivb}	15	quoniam	20_a	14
scalenis	B_{ia}	18	Scalenis	20_b	16
quod	B_{ia}	20	q̃d	20_b	16
causam	B_{iia}	29	caussam	21_b	17
per	B_{iib}	32	ꝑ	22_a	15
aequalib̃	B_{iva}	4	aequalibus	23_a	22
n.	C_{iib}	29	enim	26_a	5
ac	C_{iiia}	12	at	26_a	25

There is the usual inconsistency of the use of i and j, u and v peculiar to the Latin of that time.[48]

Figures differ in most cases in size only; exceptions are:[49]

Fig. 9 in R is the reverse of Fig. 9 in N.

Fig. 13 in R is the reverse of Fig. 13 in N.

Fig. 17 in R has chord BC and arc BC; Fig. 17 in N has chord BC only.

In both treatises upper case letters are used for marking figures; in R, however, lower case letters are used in text proper with references to figures. (This also holds for the use of Roman numerals.)

The figure to the right which is the last figure in R is missing in N.

Duplicate figures in N are Figs. 10 and 11; in R, Figs. 3, 10, 14, 15, and 18.

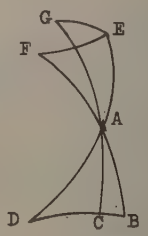

Fig. 44

48. "Between 1559 and 1562 Peter Ramus [the outstanding anti-Aristotelian of France in the sixteenth century] recommended the characters j and v to represent the consonant sounds, that up to that time had been included in i and u and were subject to confusion with the vowels. They were in consequence, for a long time called the 'Ramus consonants.' His priority in this distinction is conceded by all his contemporaries from Freigius and Nacel, who enthusiastically praise the step, to Scaliger who considers it foolish and vexatious" (F. P. Graves, Peter Ramus and the Educational Reformation of the Sixteenth Century [New York: Macmillan Co., 1912], p. 124).

49. Figures in texts are not numbered.

Since the tables of R have one more place of accuracy than those of N, all numerical references have a consistent difference. For example:

R	N
...assumpta est 2000000 circulis.	...assumpta est 200000 circulis.

Other textual differences are:

R	Page	Line	N	Page	Line
...ubi propter angulum e datum...	B_{iib}	29	...ubi propter angulum c datum...	23_a	11
...erit par ratio subtensae dupli id, ad subtensam ik.	B_{iva}	24	...erit par ratio subtensae dupli ID, ad subtensam dupli HI, atq subtensae duplicis BI ad subtensam IK.	23_b	7
...ut in sequentia figura...	B_{ib}	14	...ut in altera figura...	21_a	15
...circulorem sectionem...	C_{iiia}	5	...circulorum sectionum...	26_a	17
...nempe aequalia ipsi aeb...	C_{iiia}	7	...nempe aequalia ipsa AEB...	26_a	19
Missing			Datur igitur FG in iisdem partibus quibus etiam FC data est.	26_a	24
...quorum expectebatur demonstratio...	C_{ivb}	13	...quarum expectebatur demonstratio...	26_b	25
Missing			Haec obiter de Triangulis, prout in instituto nostro fuerint necessaria modo sufficiant. Quae si latius tractari debuissent, singulari opus erat uolomine.[50]	27_a	18

On page 54 in the 1873 edition of De revolutionibus orbium coelestium, which shall be designated by T, there is a note which states that "datis" of line 2 is "datus" on page B_{ia} line 32 of R; this statement is incorrect.

There is also a note on page 55 in T which says that "ad subtendentem circumferentiam, per quam" in line 25-26 is "...per quem" in N; there is no mention of its being in R, however, R also has "...per quem."

Page 59 in T has the statement that line 11 has "duplam" while N has "duplum," although it is not mentioned, R also has "duplum."

T does not mention the second of the listed differences.

The following is an illustration of the tables in the trigonometric treatise:

50. See p. 52.

CANON SVBTENSARVM IN CIRCVLO RECTARVM LINEARVM

		0	1			0		1	
1	2909	2909	177433	. 59	31	90174	2909	264677	. 29
2	5918		180341	. 58	32	93083		267585	. 28
3	8727		183250	. 57	33	95992		270493	. 27

This is a table of sines for the radius equal to 10000000 for the interval of one minute. Rheticus intended that the cosines should also be read from this table. The columns without headings are the difference columns.

This is an excerpt from the tables in the astronomical work.

Canon subtensarum in circulo rectarum linearum							
Circum-ferentiae		Semisses dupl.circumferen.	Differentiae	Circum-ferentiae		Semisses dupl.circumferen.	Differentiae
Pt.	scr			Pt.	scr		
0	10	291	291	6	10	10742	289
0	20	582			20	11031	
0	30	873			30	11320	

This table of sines is computed for the radius equal to 100000 for the interval of ten minutes. The abbreviations for "partes" and "scrupulum" are comparable to the degrees and minutes of modern tables. Here as elsewhere in his text, Copernicus differs from Regiomontanus in that he uses the descriptive form of the sine function, namely, half the chord of twice the arc or angle.

Because of the ambiguity of the literal translation of the work of Copernicus, it is necessary that an explanation of his trigonometric methods accompany most of the theorems.

The first theorem of Copernicus pertains to a plane triangle: "The sides of a triangle with three given angles are given."

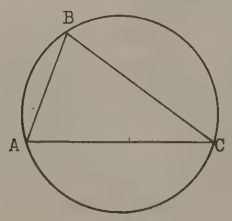

Fig. 45(a)

Let any triangle ABC be circumscribed by a circle according to Book IV, 5, of the Elements of Euclid. Therefore AB, BC, CA will be equal to the circumference in the way 360 parts are equal to two right angles. Now given the circumference, the sides of the triangle, inscribed in the circle as chords, are also given by the table where the diameter is assumed to be 200000.

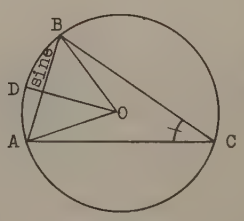

Fig. 45(b)

An inscribed angle is measured by one-half its intercepted arc and the sine of an angle is equal to one-half the chord of this arc.

In general it is not possible to compute the sides of a triangle knowing only the angles; Copernicus assumes a knowledge of the diameter which in this case is not a specific measurement, hence what he really should state is that the ratio of the sides of a triangle will be known if the angles are known.

Copernicus's proof implicitly contains the Sine Law for plane triangles. It will be recalled that the theorem relating to the sides of a triangle when the angles are given occurred in both the first and second books of De triangulis libri quinque of Regiomontanus. In the proof in the second book, Regiomontanus also refers to the Sine Law:

> This proposition presents no difficulties unless you negligently disregard the fact that any two sides of a triangle will be in the same ratio as the sines of the angles opposite them, which is plainly evident to be the converse of what has been said before.

In the second theorem, Copernicus states two cases for the solution of plane triangles: "If two sides of a triangle are given together with one of the angles, the remaining angles and the remaining sides will be known. For the given sides are either equal or unequal. The given angle is either right, acute or obtuse. The given sides include the given angle or they do not include it."

In the proof of Proposition 2, Copernicus considers an isosceles triangle in which the two equal sides and the angle included by them are known. The remaining cases he puts into the third, fourth, fifth, and sixth proofs. In the third he considers a right angle included between two unequal but known sides; the acute angle is considered in the fourth, while the fifth treats the case where the angle is obtuse; and in the sixth Copernicus considers a triangle in which a given angle subtends one of the given sides.

Theorem 7 treats of the case of a triangle with three given sides. Here there is a great similarity to the work of Regiomontanus.[51] Copernicus' theorem is: "If all the sides of a triangle are given, the angles will also be given."

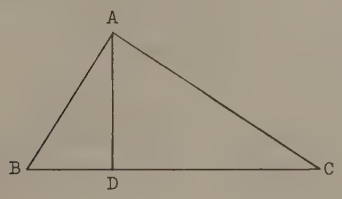

Fig. 46

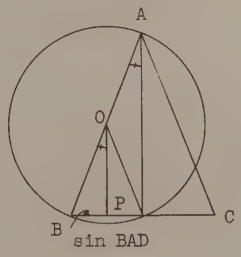

Fig. 47

It is too well known to be worth mentioning that each angle of an equilateral triangle is one-third of two right angles. It the theorem is also clear in the case of an isosceles triangle; for each of the equal sides is to the third side as the half of the diameter is to the side subtending the arc by which the angle included by the equal sides is given according to the table wherein the 360 parts around the center are equal to four right angles. Therefore it now remains to demonstrate this in the case of scalene triangles, which we divide in the usual way into right triangles. Therefore let there be the scalene triangle ABC whose sides are known; upon the side which is the longest, namely, BC, let the perpendicular AD fall. Now Book II, 13, of Euclid tells us that the square on AB which subtends the acute angle is less than the square on the other two sides by twice the product of BC and CD. Now it is necessary for angle C to be acute, for otherwise

51. See p. 24.

Figure 47 pertains to the case of an isosceles triangle. Copernicus is considering only half of the figure, because the perpendicular divides an isosceles triangle into two congruent triangles. The diameter of which he speaks is the diameter of the circle circumscribing one of these triangles which will be AB or AC. Therefore he states implicitly that

$$AB/BC = \tfrac{1}{2}AB/BD$$

The formula that Copernicus is using in the case of the scalene triangle is:

$$\overline{AB}^2 = \overline{AC}^2 + \overline{BC}^2 - 2\,\overline{BC}\cdot\overline{CD},$$

which he implicitly changes to

$$\overline{CD} = \frac{\overline{AC}^2 + \overline{BC}^2 - \overline{AB}^2}{2\,BC}$$

AB would be the longest side which, according to Book I, 17, and the two following propositions of Euclid, is contrary to the hypothesis. Therefore BD and DC are given, and there will be the right triangles ABC and ADC with sides and angles given - as has so often happened before - and so the angles of triangle ABC which were sought have become known.

Another Way

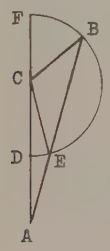

Fig. 48(a)

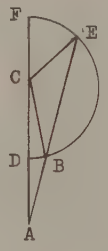

Fig. 48(b)

Similarly, the next to the last proposition in Book II of Euclid will perhaps give us an easy method. With BC the shortest side as radius and with point C as center, we describe a circle which will cut either one or both of the remaining sides. First let it cut both: AB at the point B and AC at D; and let the line ADC be extended to point F in order to complete the diameter DCF. With this construction it is clear from the proposition of Euclid that the product of FA and AD is equal to the product of BA and AE, since each is equal to the square of the line which drawn from A touches the circle. But the whole AF is given, since all its segments are given, namely, CF, CD, which are drawn from the center to the circumference and are equal to BC and AD which is the difference between CA and CD. Wherefore since the product BA and AE is given, the length of AE is also given, and so is the remainder, BE subtending arc BE. By joining EC we shall have the isosceles triangles BCE with all its sides given. Therefore the angle EBC is given. Hence, in the triangle ABC, the remaining angles C and A may be learned from the foregoing demonstration.

However, let the circle not cut AB as in the other figure, then AB falls upon the concave circumference. Nevertheless BE will be given. In the isosceles triangle BCE, angle CBE will be given and also the exterior angle ABC. By the previous method the remaining angles are determined.

and a similar expression for BC. These are the same equations that Regiomontanus used.

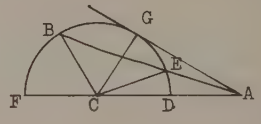

Fig. 49

Consider Fig. 49 which is constructed in a manner similar to Fig. 48(a), and verify that $FA.DA = \overline{CA}^2$.

$$\overline{GA}^2 = \overline{CA}^2 - \overline{CG}^2$$
$$\overline{GA}^2 = \overline{CA}^2 - \overline{CD}^2$$
$$\overline{GA}^2 = (CA - CD)(CA + CD)$$
$$\overline{GA}^2 = DA.FA$$

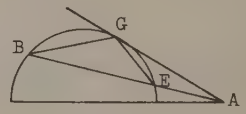

Fig. 50

Consider Fig. 50 and verify that $BA.EA = \overline{GA}^2$.

Triangles GBA and GEA are similar, hence their sides are proportional so

$$BA/GA = GA/EA \text{ or } \overline{GA}^2 = BA.EA$$

In his consideration of plane triangles, Copernicus does not give the solution of a triangle when two angles and a side are known.

In his statement of the Sine Law for spherical triangles, Copernicus imitates Regiomontanus[52] by considering the theorem for a right triangle, however, he does not

52. See p. 28.

also state it for a general triangle. His theorem is: "In spherical triangles having a right angle, the chord subtending twice the side opposite the right angle is to a chord subtending twice one of the sides forming the right angle as the diameter of a sphere is to the chord which subtends twice the angle included in the great circle of the sphere by the first side and the remaining side."

For let there be a spherical triangle ABC, of which angle C is right. I say that the chord subtending twice AB is to the chord subtending twice BC as the diameter of the sphere is to the chord which subtends twice the angle BAC on the great circle of the sphere.

With A as a pole let DE, the arc of a great circle, be described, and let ABD and ACE, the quadrants of the circle be completed. And from the center F of the sphere, let the common sections of the circles be drawn: FA is the common section of the circles ABD and ACE; FE, of circles ACE and DE; and FC, of the circles AC and BC. Then let BG be drawn at right angles to FA, then join the points G and I.

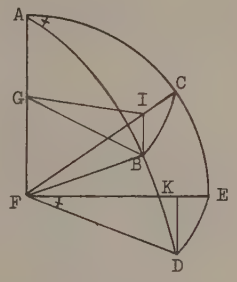

Fig. 51

The proof that Copernicus gives is rather involved, but the following arguments adhere as closely to his text as clarity of exposition will allow:

Angle AED is a right angle, because AE passes through the pole of DE.

Angle ACB is a right angle by hypothesis.

Planes EDF and GIB are perpendicular to plane AEF by construction; hence, planes EDF and GIB are parallel.

KD is perpendicular to plane AEF, because it is perpendicular to the line of intersection of the plane in which it lies and plane AEF.

BI is perpendicular to the plane AEF for the same reason; hence KD and BI are parallel.

BG is parallel to FD because angles FGB and GFD are right angles.

Angles FDK and GBI are equal, because they are angles in parallel planes with their arms respectively parallel.

But angles FKD and GIB are right angles; hence, triangles GBI and FKD are similar; so DK/BI = DF/BG.

BI is the sine[53] of CB.

BG is the sine of BA; DK is the sine of angle A; and DF is half the diameter of the sphere; hence,

$$\frac{\sin A}{\sin CB} = \frac{\text{semidiameter}}{\sin AB} .$$

In his fourth theorem pertaining to spherical triangles, Copernicus uses the proof of the preceding theorem. The fourth theorem is: "In any triangle having a right angle, if another angle and any side are given, the remaining angle and the remaining sides will be given."

Here again Copernicus treats the three possible arrangements of sides and angles. He constructs the accompanying figure in the following manner: With C as a pole, let the arc DE of the great circle be described; let the quadrants CAD and CBE be completed; and let AB and DE be produced, until they intersect at F.

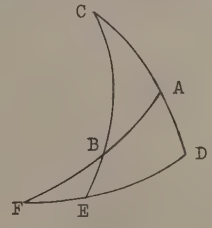

Fig. 52

53. Copernicus uses the expression "BI is half the chord of twice the arc."

In his treatise on triangles,[54] Christopher Clavius[55] points out an error in this proposition of Copernicus. Clavius maintains that: "In a right spherical triangle, if one of the angles which is not a right angle is given with an arc adjacent to the right angle then the other oblique angle and the two remaining sides can be determined." It will be recalled that in stating this theorem Regiomontanus also required that the spherical right triangle have two oblique angles.[56]

The fifth theorem on spherical triangles is: "The sides of a right triangle, of which the angles are given, are also given."

Let the preceding diagram be kept. Because angle C was given, the arc DE and EF, the remainder of the quadrant, are given. And since BEF is a right angle, because BE passes through the pole of EF, angle EBF is equal to its vertical angle which was given. Therefore the triangle BEF which has the right angle E and the angle at B together with the side EF given, has all its sides and angles given according to the preceding theorem. Therefore BF is given, and so is AB, the remainder of the quadrant. Similarly in triangle ABC, the remaining sides AC and BC are known.

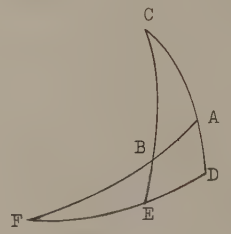

Fig. 53

Angle ECD = arc ED.
FE = 90° - ED.
CE is perpendicular to FE.
Angle FBE = angle CBA.
Hence by applying the preceding theorem, triangle BFE is completely solved.
Therefore FB is known.
Hence BA is known because BA = 90° - BF.
In a similar manner CB and CA can be determined.

The next three theorems of Copernicus are similar to Propositions 35 and 55 of the third book of De triangulis libri quinque.[57]

Theorem VI: If in the same sphere two right triangles have another angle equal to another and one side equal to one side, the sides may be adjacent to the equal angles or lie opposite one of the equal angles, they will have the remaining sides equal and the remaining angles equal.

Theorem VII: Now even if there is no right angle, but provided that the sides which are adjacent to the equal angles are equal to one another, the same thing will be shown.

Theorem VIII: Now, further, if two triangles have two sides equal to two sides and an angle equal to an angle, either the angle which the equal sides include, or an angle at the base, they will have base equal to base and the remaining angles equal to the remaining angles.

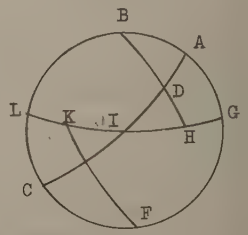

Fig. 54

54. Christopher Clavius, Opera (Mayence: A. Hierat, 1611), vol. II, p. 199.
55. Christopher Clavius was born in Bamberg in 1537. He entered the Jesuit order, studied in Coimbra, and afterwards became professor of mathematics at the Jesuit College in Rome. At the request of Gregory XIII, Clavius undertook the reform of the calendar, which he completed in 1582. Clavius died at Rome in 1612 (J. C. Poggendorf, Biographische-Literarische Handwörterbuch für Geschichte der Exacten Wissenschaften [Leipzig: Johann Barth, 1863], vol. i, p. 458).
56. See p. 30, prop. 27.
57. See p. 27.

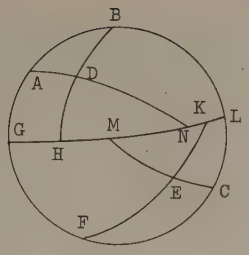

Fig. 55

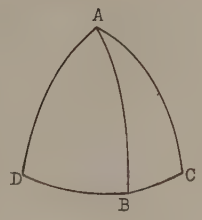

Fig. 56

Concerning the above theorem Clavius says: "This is not true, unless we add that both of the angles at the base are either larger or smaller than a right angle."[58] Clavius considers the triangles ADB and ADC; AD and AB are equal to AD and AC; D is common to both triangles. The hypothesis of the theorem is fulfilled, yet DB and DC are not equal.

This error must have originated with Copernicus for Regiomontanus states explicitly at the end of his proposition:[59] "...if the triangles are to be equilateral and equiangular, it is necessary that both the other angles be either acute or obtuse."

The next two theorems are also found in the work of Regiomontanus.[60]

Theorem IX: Moreover, in isosceles spherical triangles, angles at the base are equal.

Theorem X: If two triangles in the same sphere have the sides of one severally equal to the sides of the other, they will have the angles of one severally equal to the angles of the other.

In Theorem XI which states: "Every triangle which has two sides and an angle given will have all the sides and angles given." Clavius maintains that Copernicus is again in error.[61]

...For although AD and AB with angle D are known, yet not withstanding we arrive at the knowledge of other sides and other angles than would be expected, for the remaining side is able to be either DB or DC, etc. It is necessary to establish beforehand which remaining side and angle are to be determined.

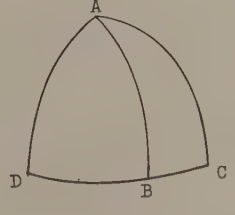

Fig. 57

In a modern text, this theorem is known as one of the ambiguous cases for the solution of spherical triangles. The possibility of more than one solution was noticed by Regiomontanus,[62] hence, he did not make the error that Copernicus did.

Copernicus continues with the cases for the solution of oblique spherical triangles.

Theorem XII: Furthermore if any two angles are given together with one side, there will be the same results.

Again Copernicus does not recognize the possibility in the number of solutions that can arise; Clavius[63] makes a note of this oversight in his trigonometric treatise.

Theorem XIII: All the sides of a triangle being given, the angles are also given.

58. Op. cit., p. 181.
59. See p. 27, prop. 45.
60. See text, pp. 85 and 91 in 1533 ed., pp. 76 and 82 in 1561 ed.
61. Loc. cit.
62. See p. 30.
63. Loc. cit.

The following is another theorem almost identical with one of Regiomontanus:[64] "If a given arc of a circle is cut anywhere so that both of the segments together are less than a semicircle, and if the ratio of half the chord subtending twice one segment to half the chord subtending twice the other segment is given, the arcs of the segments will also be given."

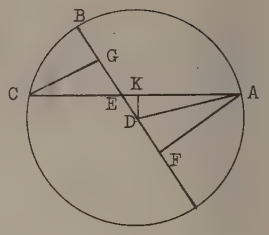

Fig. 58

The last theorem of Copernicus is: "If all the angles of a triangle are given, and though none is a right angle, all the sides will be given."

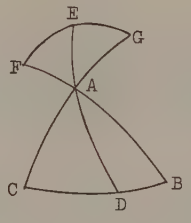

Fig. 59

Let there be the triangle ABC, all the angles of which are given, but none is a right angle. I say that all the sides are given too. For from some one of the angles, say A, draw the arc AD passing through the poles of CB. AD will intersect BC at right angles, and AD will fall within the triangle, unless one of the angles at the base, angle B or angle C, is obtuse and the other is acute. If that were the case, the arc would have to be drawn from the obtuse angle to the base. Therefore with the quadrants BAF and CAG completed and with B and C as poles, let the arcs EF and EG be described. In these right triangles, half the chord subtending twice arc AE is to half the chord subtending twice arc EF as half the diameter of the sphere is to half the chord subtending twice the angle EAF. Similarly, in triangle AEG having a right angle at G, the half of the chord subtending twice AE is to half the chord subtending twice EG as half the diameter of the sphere is to half the chord subtending twice angle EAG. Since these ratios are equal, half the chord subtending twice EF is to half the chord subtending twice EG as half the chord subtending twice angle EAF is to half the chord subtending twice angle EAG. Since the arcs FE and EG are given, for they are complements by which angles B and C differ from right angles, therefore, we shall have by means of them the ratios of sines of the angles EAF and EAG, i.e., the ratios of angles BAD and CAD, which are their vertical angles. Now the whole angle BAC has been given; therefore, by the preceding theorem the angles BAD and CAD will also be given. Then by the fifth theorem we shall attain sides AB, BD, AC, and CD, and the whole of arc BC.

In triangle AEF:

$\sin AE/\sin EF = \text{radius}/\sin EAF$

In triangle AEG:

$\sin AE/\sin EG = \text{radius}/\sin EAG$

Hence,

$\sin EF/\sin EG = \sin EAF/\sin EAG$

Therefore,

$\cos B/\cos C = \sin BAD/\sin DAB$

This formula is identical with the one of Regiomontanus.[65]

The conclusion drawn from this analysis of the trigonometry of Copernicus is that his treatise is consistent, precise, and worthwhile. While it lacks many of the details of the work of Regiomontanus and reveals the failure of Copernicus to recognize the possibility of more than one solution in two of the cases of the solution of

64. See p. 30.
65. See p. 30.

spherical triangles, still it manifests a condensation of the principal propositions of De triangulis libri quinque. Copernicus did not consider his treatise an exhaustive one, for he said himself at the end of the last theorem:

> This which has been said concerning triangles, according as they are necessary for our undertaking, will be sufficient. For if they had to be treated more in detail, the whole would be of unusual size.

Since, to all appearances, Copernicus has mastered the contents of De triangulus libri quinque, the question naturally arises: "From whom could Copernicus have received his information concerning the trigonometry of Regiomontanus?"

There is nothing original about the attempt to ascertain whether Copernicus acquired his trigonometric knowledge independently of the treatise of Regiomontanus, or if he is indebted to the work of this German scholar in at least a small degree. Reasons have been produced for and against both possibilities. Before attempting to strengthen the opinion either way, it is desirable to state what results others have attained as well as the evidence and interpretation that led to the conclusions arrived at in the present work.

Leopold Prowe[66] makes two conjectures, the first of which is that Copernicus wrote his trigonometrical treatise independently of the work of Regiomontanus. His source for this opinion is the dedicatory epistle of Rheticus to George Hartmann in De lateribus et angulis triangulorum.

> ...Only recently has the treatise of Regiomontanus become accessible; but before he was able to see the work, the famous and learned Dr. Nicolaus Copernicus, whilst he elaborated the doctrine of motion based on Ptolemy, wrote this excellent treatise on triangles. I know you will have great admiration for this work, when you see the many ingenious things which it contains.

The second conjecture of Prowe is that at some uncertain time, Copernicus became acquainted with the trigonometry of Regiomontanus. He based this opinion on a statement from "Ad candidum lectorem," written by Valentine Otho in the posthumous edition of the work of Rheticus.

> Copernicus, the wonder of our time, imitated him Ptolemy ; Thus for instance, Ptolemy gathered what was important for his purpose from Theodosius and Menelaus, so similarly Copernicus borrowed directly, with ingenuity and brevity, from Ptolemy and Johann Regiomontanus, those things which he considered necessary for his work.

In Chapter IX of his monumental work on Copernicus, Birkenmajer[67] summarizes his arguments to prove that the Copernican trigonometry was an independent treatise.

> Copernicus became acquainted with the trigonometry of Regiomontanus only after 1533, possibly in 1539, when Rheticus brought to Ermland a copy of the work of Regiomontanus and presented it to Copernicus with other books.
>
> In Cracow he listened to lectures, i.e., to the commentaries of Brudzewo, who already used trigonometry in its modern form, i.e., with the sine and not chords as did Ptolemy.
>
> Copernicus took the chief basis of his trigonometry from the Almagest, known to him very early from the excerpts made by Peurbach, Books I-VI and by Regiomontanus, Books VII-XIII of the Epitome. He used the sine instead of the chord, but not under this name as Regiomontanus did in his trigonometry (but not in his Epitome), but in the descriptive form - half the chord of double the arc. In this he followed an old form of the twelfth century translation of Geber, where this expression appears without the name sine. From Geber's Ptolemy, which he became acquainted with in Cracow, Copernicus took the astronomical notion of the length of the equator accepted by him in the Commentariolus, and from this book he also took an important phase of spherical trigonometry used by Regiomontanus from the same source.

66. Op. cit., pp. 482-489.
67. Mikołaj Kopernik (Cracow: Ksiegarni, 1900), p. 235.

It is important to state when Copernicus' trigonometry was written. It must have been formulated before his first heliocentric treatise, not later than 1512. If we consider that this work, in which trigonometry and its tables were only tools, must have been in progress for several years, we come to the conclusion that the mathematical part must have been completed in the first years of the sixteenth century.

Elsewhere in this chapter, Birkenmajer states that neither Montucla, Kästner, nor Cantor give satisfactory evidence that Copernicus borrowed his trigonometrical ideas from Regiomontanus.[68]

Braunmühl's statements regarding this question are:[69]

The thirteenth chapter of Revolutions contains plane trigonometry that on the whole offers nothing noteworthy on the fundamental problem of the triangle, solved according to the method of the Greeks by dividing it into two right-angled triangles; only everywhere the half chord is used instead of the whole chord. Is it possible to see in this evident avoidance of the word sine, a connection with Werner, whose book on triangles we know was in the possession of Rheticus?

In Chapter XIV, Copernicus considers spherical trigonometry, wherein he places the Sine Law, or rule of four quantities as the most important proposition. From whom had the master learned this theorem? Possibly from the verbal communication of Rheticus who in turn had received it from Werner? Among the existing possessions of Copernicus, there is a copy of the well known edition of the nine books of Geber which were printed in 1534, as well as a copy of the De triangulis of Regiomontanus with a manuscript dedication of Rheticus, which unfortunately is not dated, so that no one can decide when this work was given to Copernicus. Possibly his knowledge of the Sine Law comes from one of these works.

There is nothing in the Commentariolus to show the status of the trigonometry of Copernicus when this work was written. In this treatise Copernicus remarks:[70]

However, I have thought it well, for the sake of brevity, to omit from this sketch all mathematical demonstrations, reserving these for my larger work. But in the explanation of the circles I shall set down here the lengths of the radii; and from these the reader who is not acquainted with mathematics will readily perceive how closely the arrangement of circles agrees with the numerical data and observations.

A portion of the preface of the Opus palatinum de triangulis offers some interesting information:

...When George Joachim Rheticus considered this [trigonometry of Regiomontanus,] he began to consider how he could enrich the doctrine and bring about its completion. While he was making these considerations, the news of the hypothesis of the admirable Copernicus was spread abroad. Although Rheticus was engaged as a professor of mathematics at the flourishing Academy of Wittenberg, still he could not be satisfied until he visited this great man at his first opportunity. So he went to Copernicus. When he became personally acquainted with the hypothesis of [Copernicus], he resigned his professorship and stayed with him. At that time Copernicus was directly occupied with his work, Revolutions. He had just completed the doctrine of the second motion, so nothing more remained except the doctrine of the first motion. Copernicus decided to leave this completely untouched, but his friend urged him to such an extent that he could not honestly refuse him. Only when he was unable to seek aid from books, of which he had so few, as I heard from Rheticus, did he ponder upon the invention of suitable demonstrations for his theories. However, after much deliberation, he was still unable to fulfill his desires. In sheer desperation of ever inventing the necessary demonstrations, he decided to suppress his work. Still, revering his friend, whose theories he considered authoritative, [Copernicus] reconsidered the thoughts which he had rejected and at last invented the demonstrations which he had so greatly desired.

68. Ibid., p. 220.
69. Op. cit., p. 141.
70. Edward Rosen, op. cit., p. 59.

The following is a quotation from Rheticus' dedicatory epistle in De lateribus et angulis triangulorum:

> ...So he [Copernicus] consented that I publish it [trigonometric treatise] now, considering that we need the doctrine of triangles in the exposition of Ptolemy; and to you [George Hartmann] I have dedicated it.... And if anyone profits by my work in this form of public enterprise, to the usefulness of which our studies contribute, I wish that one to attribute thanks to my teacher.

In Chapter II of this thesis,[71] it has been stated that Rheticus came into the possession of the trigonometry of Werner in 1542, when he paid a visit to George Hartmann. He began the publication of this treatise in 1557, but never finished it. A quotation from the preface of this edition emphasizes the fact that Rheticus did not have Werner's book with him when he went to Ermland. In this preface Rheticus says:

> After Werner's death, George Hartman, a German mathematician, preeminent in virtue and learning, forthwith collected these books, De meteorscopiis and De triangulis sphaericis, as one gathering planks which have been scattered in different directions after a ship wreck, and these materials which had been, as it were, inscribed on Sybil's leaves and scattered abroad he brought together and gave them to me fifteen years ago.

From the foregoing quotations, the following conclusions have been drawn. There is no reason to suppose, as Birkenmajer does, that the mathematical demonstrations withheld from the Commentariolus were the same as those Copernicus published in De lateribus et angulis triangulorum and De revolutionibus orbium coelestium.

Since Rheticus was not in possession of Werner's De triangulis sphaericis until 1542, and began in 1557 the publication which he never finished, it is evident that, contrary to Braunmühl, Copernicus did not see this manuscript. However its contents could have been given verbally to Copernicus by Rheticus, because it is not certain that Rheticus saw this work for the first time in 1542, since the friendship between him and George Hartman, who obtained possession of the manuscript after Werner's death, antedated that year. It might be noted here also that Werner did use the sine function under that name.[72]

In the De revolutionibus orbium coelestium, Copernicus makes no mention of Rheticus. An explanation for this has been given, namely, that the difference in their religious beliefs caused Copernicus to omit the name of Rheticus from his work, not for personal reasons, but to offset any religious animosity to either of them that might arise from other sources. Explicitly, Rheticus claims no credit for the success of any endeavor of his in causing the publication of the trigonometric treatise of Copernicus. The seeming cautiousness of his teacher or his own loyalty and unselfishness might have prompted this reserve. However in the dedicatory epistle of De lateribus et angulis triangulorum, there is an implication that he considered his work a contribution to public service. What else he meant can only be conjectured.

Rheticus was thoroughly acquainted with the trigonometry of Regiomontanus before he went to Ermland. He foresaw great possibilities in the completion of the theory of triangles. The sponsoring of the publication of the trigonometric portion of De revolutionibus orbium coelestium and the beginning, fifteen years later, of the editing of Werner's De triangulis sphaericis, besides the contributing of his own researches to the theory of trigonometry show wherein his interests primarily lay.

Since, on the one hand, Valentine Otho states that Copernicus borrowed from Ptolemy and Regiomontanus, while on the other, Rheticus, who had mastered the trigonometry

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71. See p. 36.

72. "Liber Secundus, Def. II: Sinus rectus datae sectionis est ipsius geminatae semibasis."
"At semibasi et sinu recto vocabulis indifferentur posterius utar, eo quod significatur partum distent."

of Copernicus before his visit to Ermland, maintains that Copernicus did not have access to the trigonometric work of Regiomontanus before that visit; since Copernicus sought in vain for mathematical demonstrations from the books which were "only too few" in his own library and only invented the necessary demonstrations after much urging on the part of Rheticus, whose theories Copernicus considered authoritative; and since the trigonometry of Copernicus admits a marked similarity to that of Regiomontanus, then no matter what status the trigonometry of Copernicus had before 1539, it was further developed by the information that Copernicus received from Rheticus concerning the trigonometry of Regiomontanus.

7. TRIGONOMETRY OF RHETICUS

It will be recalled that Rheticus left Ermland to superintend the printing of the astronomical and trigonometrical works of Copernicus. Scarcely was this task completed when death claimed his teacher, so Rheticus did not return to Ermland but remained in Germany to continue his work in mathematics. Valentine Otho in the "Ad candidum Lectorem" of the Opus palatinum de triangulis, says:

> ...he [Rheticus] still observed the triangle with a right angle, the principle of the whole of mathematics...he produced the calculations which completed the rules.

The Opus palatinum de triangulis is a work of two volumes, the first of which contains four treatises on trigonometry - three are due to Rheticus and one to Otho; tables of the six trigonometric functions make up the entire second volume.

Because of the importance of certain definitions in the second treatise of the first volume, it will be considered first.

"DE TRIQUETRIS RECTARUM LINEARUM IN PLANITIE"

The contents of this treatise are given in the "Scopus:"

> There are three angles and three sides in a rectilinear triangle in a plane; if three of these parts are given, then the remaining three parts will be obtained from the ratio of the given parts.

There are thirty items under the caption "Definitions and Hypothesis." The following ones have been selected because they are peculiar to the trigonometry of Rheticus.

> The perpendicular of an arc is half the chord of twice the arc.
>
> The base of the arc is half the chord of twice the complement of the arc.
>
> A quadrant is formed when the perpendicular and base intersect at the center of the circle.
>
> In every triangle with a right angle, the side which subtends the right angle is called the hypotenuse.

The three following definitions are given interpretatively instead of literally because this method affords a clearer meaning than would otherwise be obtained.

> If AB is the radius or "sinus totus," then the perpendicular is the sine and the base is the cosine. This is a triangle of the first series.

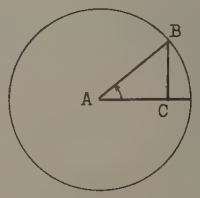

Fig. 60

If AC is the radius, then the perpendicular is the tangent and the hypotenuse is the secant. This is a triangle of the second series.

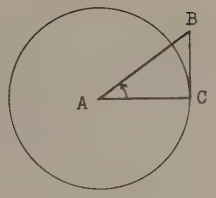

Fig. 61

If BC is the radius, then the base is the cotangent and the hypotenuse is the cosecant. This is a triangle of the third series.

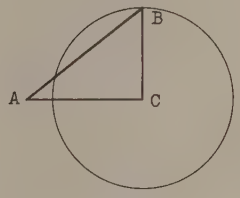

Fig. 62

The nature of the propositions of this section can be inferred from the "Scopus." "Praeceptum Octavum" has been chosen as representative. "In any triangle where all the angles are acute or one is obtuse, if two sides with one angle are given, and the sides do not include the angle, then the remaining side and angle will be known from the 'Aquisition' to the first lemma."[73]

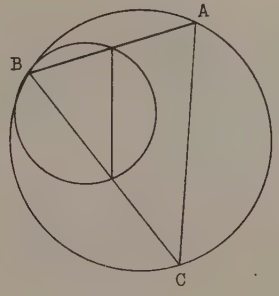

Fig. 63

If ABC is any triangle whose two sides AB and BC are given; the given angle being angle BAC, not included by the two sides, I say by the 'Aquisition' of the first lemma that the remaining angles A and C with the side AC will also be known, for in a right lined triangle in a plane, the ratio of the perpendicular of the angles is equal to the ratio of the sides which they subtend. Thus the side BC to the side AB of the proposed triangle is the same as the perpendicular of the given angle BAC[74] to the perpendicular of the required angle ACB. The first three quantities are given; wherefore the perpendicular of angle ACB is given, hence angle ACB, an angle of the first series, is given. But BAC and ACB subtracted from two right angles leaves the third angle ABC. Likewise the side AC will be known by constructing a similar ratio.

BC/AB = sin BAC/sin ACB,
thus sin ACB can be determined;
therefore angle ACB is known.

ABC = 180 - (ACB + BAC)

sin BAC/sin ABC = BC/AC
sin ACB/sin ABC = AB/AC

Therefore one side can be determined by two different ratios.

73. "In a right-lined triangle in a plane, the ratio of the perpendicular of the angles is equal to the ratio of the subtending sides."
74. The text reads, "...ita perpendiculum ACB anguli dati."

Now the ratio of the perpendicular of the angle BAC to the perpendicular of the angle ABC is the same as the side AB to the side AC, or the perpendicular of the angle ACB to the perpendicular of the angle ABC is the same as the ratio of the side BC[75] to the side AC. Hence when both ratios of three magnitudes are given, the remaining side will be determined by two ratios. In a similar manner a ratio can be set up for sides and an angle, as long as the given angle is not included by the two sides.

If a triangle has an obtuse angle follow the same theory and the remaining angles and side will be determined.

"DE FABRICA CANONIS DOCTRINAE TRIANGULORUM"

This treatise is composed of three books, the first of which contains nine lemmas; the second, ten propositions; and the third, primarily calculations of tables.

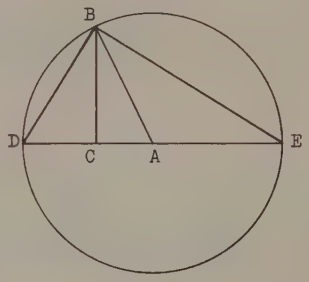

Fig. 64(a)

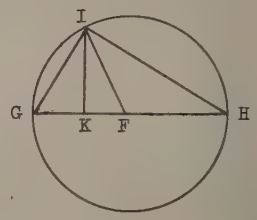

Fig. 64(b)

Several of the lemmas of Book I will demonstrate the nature of this section of the Opus palatinum de triangulis.

Lemma I: In unequal circles the ratio of the diameter of one circle to the diameter of another circle is the same as the ratio of the chords, perpendiculars, and bases of the one circle to the chords, perpendiculars and bases of similar arcs of another circle, or chords, perpendiculars, and bases of the arcs of one circle taken in any order are equal in ratio to the chords, perpendiculars, and bases of similar arcs of the other circle taken in the same order."

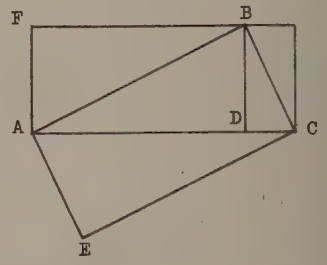

Fig. 65

Lemma III: "If from the right angle of a rectilinear right triangle, a perpendicular is drawn to the hypotenuse, the rectangular parallelogram determined by the sides that include the right angle is equal to the parallelogram formed by the hypotenuse and perpendicular."

Lemma V: "A quadrilateral whose sides are equal in magnitude to half the sides of a cyclic quadrilateral is itself a cyclic quadrilateral.

A cyclic quadrilateral must satisfy Ptolemy's theorem hence in Fig. 66, BD.CE = CD.BE. If the sides of this quadrilateral are bisected, the quadrilateral formed from these magnitudes is cyclic, because the bisection is equivalent to dividing both members of the equation by 4.

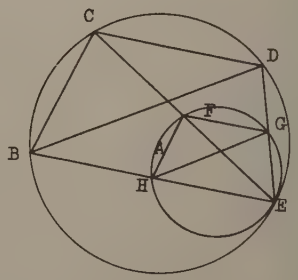

Fig. 66

From the ten propositions the two following have been selected.

"Proposition prima": The diameter of a circle is given; inscribe the sides of a triangle, quadrilateral, pentagon, hexagon, and decagon."

75. Text has AB.

Recall that this proposition, taken from Ptolemy, occurred in the works of Peurbach[76] and Regiomontanus.[77] Copernicus has it, too, but not in the trigonometric portion of his work. Compare Fig. 68 with Fig. 34.

"Proposition quinta": "The perpendiculars to two or more arcs of a circle being given, the perpendicular of their sum will also be given."

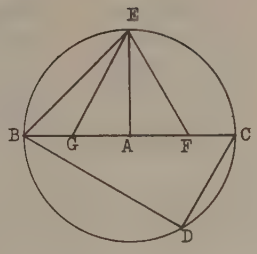

Fig. 67

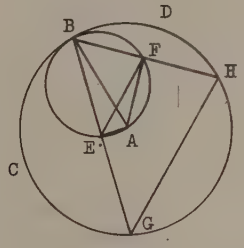

Fig. 68

With A as center describe the circle BCD. Let BC and BD be the arcs of the circle whose perpendiculars are given. I intend to give the perpendicular of the one arc CBD made up of arc BC and BD. Join A and B; with AB as diameter, describe the circle BEAF. Arc BC is half BCG and BD is half BDH.[78] BEG is the chord subtending arc BCG; BFH is the chord subtending arc BDH. Draw EA, FA, and EF. BE is the perpendicular of BC and BF is the perpendicular of BD. EA is the base of arc BC and AF is the base of arc BD. EF, which is one-half GH, is the perpendicular of arc DBC. Arc DBC is half GH, which subtends twice the arc CBD. EH is known, because in a quadrilateral if all the sides and one diagonal are given, the other diagonal will be determined by the "Aquisition" for the fourth lemma.

$BE = \sin BC$
$EA = \cos BC$
$BF = \sin BD$
$AF = \cos BD$
$EF = \sin DBC$

From the theorem of Ptolemy which Rheticus mentions as the "Aquisition" to the fourth lemma:

$$EF.BA = BE.AF + BF.EA$$

Hence, after making substitutions, assuming we have a unit circle, the equation is:

$$\sin DBC = \sin BC \cos BD + \sin BD \cos BC.$$

On the following page the table from the "Dixotomia Principiorum Exquisitorum" which is the third book of this section, is given to show that Rheticus used the function "sinus versus" and to give an idea of the arrangement of his work and the amount of labor the calculation of his tables must have entailed. In the text the two sets of data are given in parallel columns.

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76. See p. 32.
77. See p. 33.
78. The figure in the text has N instead of H.

Basis. 18. part.	951056516295153.
Sinus versus.	4894348370484 7 -- 4894348370484 7.
Quadratum.	23954645971666238210112934 09.
Quadrat. Perpen.	95491502812626025839221252809.
Quadrat.	97886967409692649660232546219.
Radix.	312868930080461.
Dimidata.	156434465040230.
	perpend. 9. part.
Perpend. 18. part.	309016994374947 -- 309016994374947.
	95491502812526025839221252809.
	156434465040230 -- 156434465040230.
	24471741852422942055518452900.
	10000000000000000000000000000000.
	97552825814757705794448154 7100.
	Basis. 9. part. 987688340595137.

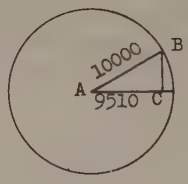

Fig. 69

$2 \sin^2 9^\circ$ = vers sine 18°
sin 9° = 15643
2 sin 9° = 31286
$4 \sin^2 9^\circ$ = 09784
$2 \sin^2 9^\circ$ = 04892
vers sine 18° = 04892

"DE TRIANGULIS GLOBI CUM ANGULO RECTO"

This section is divided into four books; the first book contains forty theorems, without proofs, pertaining to the doctrine of spherical triangles. All the theorems deal with the ratios of the trigonometric functions of the sides and angles, as is indicated by the following theorem which is representative of the other thirty-nine.

Theorem XVIII: "In a right spherical triangle, the ratio of the perpendicular of the side of the triangle which subtends the right angle, to the perpendicular of either of the sides that include the right angle is the same as the ratio of the base of the other side which includes the right angle to the base of the angle which subtends this side."

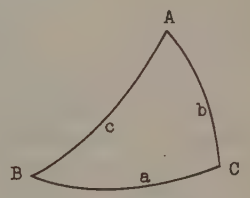

Fig. 70

The formula that Rheticus refers to is:

$$\sin c / \sin a = \cos b / \cos B.$$

If for sin a we substitute its value which we obtain by applying the law of sines, namely, sin a = sin c sin A, we arrive at the more familiar formula:

$$\cos B = \cos b \sin A.$$

The "Scopus" of Book II gives the contents of the second part of this section:

> In a spherical right triangle in which one side is less than a quadrant of a great circle, there are two acute angles and three sides besides the right angle. Of these five parts if two are given the other three will be determined.

The following example with its accompanying tabulated data was chosen to demonstrate the contents of Book II.

> In Triangulo Globi cum angvlo recto BCD, sit alius praeterea angulus datus BDC, partium XXIII, scrupulorum XXVIII, solis obliquatio maxima, cum latere BD: quod rectum subtendit angulum partium XX. Exquirenda sunt BC latus, obliquatio XX partium, cum DC latere quod tempora ostendit, cum quibus in Horizonte recto partes XX eius qui est per medium signorum ascendunt, & DBC angulus, quem cum Horizonte recto includit.[79]

79. Taken from its astronomical setting the problem is: "In the spherical triangle with the right angle BCD, let another given angle be 23° 8' and the side subtending the right angle, 20°. Determine the sides BC, DC and the angle DBC."

DATA
DE CANONE DOCTRINAE
TRIANGULORUM

	De Serie	Hypotenusa.	Per-pendiculum.	Basis.
BDC anguli	Prima	10000000000	3982154775	9172919020
part. 23	Secunda	10901655164	4341207817	10000000000
scrup. prim. 28	Tertia	25112032467	10000000000	23035064025
BD Arcus	Prima	10000000000	3420201433	9396926208
part. 20	Secunda	10641777725	3639702343	10000000000
	Tertia	2923804002	10000000000	27474774194

In Book III, Rheticus employs a spherical triangle of a more general nature, i.e., one with fewer restrictions on the magnitude of the angles and sides. He establishes a set of symbols and then tabulates his theorems in a unique way as is illustrated by the following:

TRIANGVLORVM GLOBI
DIFFERENTIAE

Ratione Angvlorvm			Ratione Latervm	
Triangvlvm Globi			Triangvlvm Globi	
	Cum duobus rectis.	R R	Cum duobus quadrantibus	Q.Q.
1	Recto.	R	Quadrante	Q.
2	Acvto.	A	Minore quadrante	Q.mi.
3	Obtuso.	O	Maiore quadrante	Q.ma.
	Cum duobus acutis	A A	Cum duob.minorib.quadr.	Q.mi.Q.mi.
4	Recto	R	Quadrante	Q.
5	Acvto	A	Minore quadrante	Q.mi.
6	Obtvso	C	Maiore quadrante	Q.ma.
	Cum duobus obtusis	O O	Cum duobus maiore quadr.	Q.ma.Q.ma.
7	Recto	R	Quadrante	Q.
8	Acvuto	A	Minore quadrante	Q.mi.
9	Obtvso	O	Maiore quadrante	Q.ma.
10	Cum recto, acuto, & obtuso.	R A O	Cum quadrante, minore & maiore quadrante	Q. Q.mi. Q.ma.Q.ma.

00	6	In Triangulo Globi cum duobus angulis obtusis, latere quae obtusos obent maiora sunt quadrantibus, sed tertius angulus cum tertio latere variat.[80]	R. Q ma. Q ma. Q mi. O. Q ma. Q ma. Q mi. A. Q ma. Q ma. Q mi. O. Q ma. Q ma. Q

There are one hundred twenty-nine theorems of the above type and arrangement which Rheticus calls "Acquisitions."

80. "In a spherical triangle with two obtuse angles, the sides opposite the obtuse angles will be greater than a quadrant, but the third angle varies with the third side."

In Book IV Rheticus established six forms of right spherical triangles. The six forms are:

First Form - sides are arcs equal to quadrants and angles are right angles.

Second Form - two sides equal to quadrants subtend right angles and include an acute angle which is opposite an arc less than a quadrant.

Third Form - two sides are equal to quadrants which subtend right angles, and include an obtuse angle which is opposite a side greater than a quadrant.

Fourth Form - sides are less than a quadrant and the two angles other than the right angle are acute.

Fifth Form - two angles are obtuse and one is right; the right angle subtends a side less than a quadrant; the obtuse angle subtend sides greater than a quadrant.

Sixth Form - angles are acute, right, and obtuse; the right subtends a side greater than a quadrant; the obtuse, a side greater than a quadrant; and the acute angle is opposite a side less than a quadrant.

Twelve theorems pertain to these six forms. Theorem III, for instance is:

"In a right spherical triangle, if the side which subtends the right angle is greater than a quadrant, and forms an acute or obtuse angle with another side, the triangle is of the sixth form."

"DE TRIANGULI GLOBI ANGULO RECTO OF VALENTINE OTHO"

From the "Scopus" of each of the five books of this section, Valentine Otho's contribution to the Opus palatinum de triangulis can be ascertained.

I. How to construct diagrams from which the doctrine of spherical triangles without a right angle can be demonstrated.

II. In all spherical triangles without a right angle, if two sides and one angle are given, the two remaining angles and the third side will be determined.[81]

III. In any triangle without a right angle, determine the third angle and the remaining two sides when two angles and a side are given.[82]

IV. Given a spherical triangle without a right angle, determine all the angles when all the sides are given.

V. Given a spherical triangle without a right angle, determine all the sides when all the angles are given.

Before an estimate of the trigonometry of Rheticus is made, it is well to recall that by 1539 Rheticus had become thoroughly acquainted with the trigonometry of Regiomontanus. In 1542 he superintended the publication of the trigonometrical treatise of Copernicus. In the same year he received from George Hartmann the De triangulis sphaericis of Werner, which he began to edit in 1557. Now Werner's treatise contained little in advance of that of Regiomontanus. Werner borrowed extensively from Theodosius, Menelaus, and the Arabs. Since the work of Regiomontanus was a challenge to the genius of Rheticus,[83] it is doubtful whether Rheticus could have been influenced by a lesser work, the De triangulis sphaericis.

Hence the conclusion reached after a careful study of the Opus palatinum de triangulis, is that Rheticus built his trigonometry on the foundation established by Regiomontanus. He discarded the traditional consideration of trigonometric functions

81. Otho considers the ambiguous case.

82. Otho considers the ambiguous case.

83. This is deduced from Valentine Otho's "Ad candidum lectorem." See quotation on p. 101.

with respect to an arc of a circle, by constructing a right triangle and making the functions depend directly on its angles. He conceived the idea of a table of secants from the hypotenuse of a right triangle.

Ramus, a French contemporary of Rheticus, had a great regard for this German mathematician. In a publication of 1569 he wrote:[84]

> ...had he [Rheticus] not been compelled to master medicine and make the practice of it sort of a Maecenas, mathematics would now for a long time be proclaiming a second Copernicus.

So the task of further developing trigonometry passed to Pitiscus and his contemporaries.

84. Edward Rosen, "The Ramus-Rheticus Correspondence," Journal of the History of Ideas, 1:368, June, 1940.

Chapter IV

PITISCUS AND HIS CONTEMPORARIES

1. INTRODUCTORY NOTE - EUROPEAN MATHEMATICS AT THE CLOSE OF THE SIXTEENTH CENTURY

At the beginning of the sixteenth century, Germany and Italy led in the study of the mathematical sciences.[1] Poland, being in close cultural contact with Western Europe and forming an integral part of it,[2] also took an active participation in scientific pursuits. As the sixteenth century neared its completion, mathematics in Germany was on the decline. Internal struggles, religious and otherwise, impeded progress in the arts and sciences, as a result of which, Kepler,[3] who in collaboration with Tycho Brahe,[4] the Danish astronomer, completed the Rudolphine Tables, was the only outstanding German mathematician for over a century.[5] Italy, however, continued to produce citizens, such as Magini[6] and Galileo,[7] who ranked among the leaders in scientific thought.

It was through the suggestion and assistance of Italy and Germany that a revival of classical studied took place in France in the sixteenth century.[8] This movement, despite its outside influence, exceeded all previous attainments[9] and was an opportunity for French genius to exert itself. Vieta,[10] the most eminent of the French mathematicians, not only made notable contributions to algebra and geometry, but originated the literal symbolism of algebra.

England concentrated most of her energy on secular matters and acquired in the sixteenth century a literature immortalized by the genius of Shakespeare and Spenser.

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1. Florian Cajori, op. cit., p. 131.
2. Avrahm Yarmolinsky, Early Polish Americana (New York: The New York Public Library, 1937), p. 7.
3. Johann Kepler was born in 1571 in Weil. He was a man of keen imagination, indomitable perseverance and uncompromising love of truth. He overcame ill-health, poverty, and misfortune, and secured a place for himself in the very highest rank of scientific men. His laws introduced order and simplicity into what would have been a chaos of observations they served as a secure basis for the splendid erection made on them by Newton. Kepler died in 1630 (Sir Oliver Lodge, Pioneers of Science [London: Macmillan Co., 1919], pp. 58-79).
4. Tycho Brahe was born in 1546. He was raised by his paternal uncle, attended the University of Copenhagen, and became interested in astronomy. He saw the new star in 1572, kept a record of his observations, and published them in 1573. When he became estranged from King Christian of Denmark, he went to Prague and sought the patronage of King Rudolf. Here he was assisted by Kepler. He died in 1601 in Bohemia (J. L. E. Dreyer, Tycho Brahe [Edinburgh: A. and C. Blake, 1890]).
5. Florian Cajori, op. cit., p. 146.
6. Giovanni Magini was born June 13, 1555 at Padua. He was professor of astrology, astronomy and mathematics at the University of Bologna in 1588. He died at Bologna in 1617 (J. C. Poggendorff, op. cit., vol. II, p. 11).
7. Galileo Galilei was one of the earliest and perhaps one of the greatest of the experimental philosophers of the modern world and the father of telescopic astronomy. He was born in Pisa in 1564 and received his early education there. He was professor of mathematics in Pisa and Padua. He died in Siena in 1642 (John Joseph Fahie, Galileo [New York: James Pott and Co., 1903]).
8. Henry Osborn Taylor, Thought and Expression in the Sixteenth Century (New York; Macmillan Co., 1920), vol I, p. 295.
9. Idem.
10. Vieta was born in Poitou in 1540 and died in Paris in 1603. He was employed throughout his life in the service of the state, under Henry III and Henry IV. He was, therefore, not a mathematician by profession, but his love for this science was very great and he devoted much of his free time to it (Frédéric Ritter, François Viète Paris, 1895).

A great scientific age followed this literary age. However, as early as 1556, the Copernican theory had found adherents in England. In Robert Recorde's[11] Castle of Knowledge this statement is made:

> ...Copernicus, a man of great learninge, of much experience and of a wonderful diligence in observation, hathe renewed the opinion of Aristarchus Samius, and affirmth that the earth not only moveth circularlye about his own center, but also may be, yes and is, continually out of the precise center of the world 38 hundred thousand miles:

Thomas Digges,[13] before 1576, appended a treatise[14] on the Copernican theory to the posthumous work[15] of his father, Leonard Digges. Concerning this work Francis Johnson says:[16]

> ...[it is] for the most part a translation into English of those Chapters of Book I of Copernicus' De Revolutionibus Orbium Coelestium in which the main features of the new heliocentric system are explained and the Aristotelian and Ptolemaic arguments against it are refuted; thus it is almost certainly the first English translation of Copernicus, and was definitely so regarded by Thomas Digges, its author.

Although the application of logarithms to trigonometry is beyond the scope of this thesis, still to emphasize Scotland's and England's contribution to the mathematics of this period, it is necessary to mention the Mirifici logarithmorum canonis descriptio of John Napier[17] and the Trigonometria of Henry Briggs.[18] The value of logarithms as a time-saving factor was early perceived and appreciated. Richard Norwood,[19] a contemporary of Briggs, wrote:[20]

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11. Robert Recorde was born at Tenby in Pembroke about 1510. He was admitted a scholar at Oxford about 1525. He graduated M.D. at Cambridge in 1545. In 1549 he was comptroller of the mints at Bristol and in May, 1551, he was appointed by the king general surveyor of the mines and money, in which capacity he served both in England and in Ireland. Recorde was practically the founder of an English school of mathematical writers (Francis R. Johnson and Sanford V. Larkey, "Robert Recorde's Mathematical Teaching and the Anti-Aristotelian Movement," Huntington Library Bulletin, 7:60-61, April, 1935).
12. Robert Recorde, Castle of Knowledge (London: R. Wolfe, 1556), p. 165.
13. "Thomas Digges was the son of an eminent English mathematician, Leonard Digges, and was born about 1545. The Dictionary of National Biography is altogether wrong in identifying him with a certain Thomas Digges or Digge, who matriculated at Queen's College, Cambridge, in 1546" (Francis Johnson, "Thomas Digges, the Copernican System and the Idea of the Infinity of the Universe," The Huntington Library Bulletin, 5:104, April, 1934).
14. "A perfit description of the celestial orbes according to the most auncient doctrine of the Pythagoreans, latelye reuiued by Copernicus and by geometrical demonstrations approued."
15. Leonard Digges, A Prognostination Euerlastinge (London: Thomas Marsh, 1576).
16. Francis Johnson, op. cit., p. 71.
17. John Napier, laird of Merchiston, was born at Merchiston Castle near Edinburgh in 1550. He devoted his leisure time chiefly to mathematics; his permanent fame rests on his mathematical discoveries. He died April 4, 1617 (E. W. Hobson, John Napier and the Invention of Logarithms, 1614 [Cambridge: University Press, 1914]).
18. Henry Briggs was born at Warley Wood, in the parish of Halifax, Yorkshire, in February, 1561. In 1577 he entered St. John's College, Cambridge, and took his M.A. degree in 1585. When Gresham College was founded in London, he became professor of geometry there. On hearing of Napier's discovery of logarithsm, he was struck with enthusiasm and went to Scotland to visit him. Briggs died at Merton College, 1630 (Thomas Smith, Vitae quorundam eruditissimorum et illustrium virorum [London: David Mortier, 1707]).
19. Richard Norwood, a teacher of mathematics and a surveyor, was born about 1590. In 1616 he was sent by the Bermuda Company to survey the island of Bermuda. He may have made subsequent visits to the islands, but according to his own statement he was for some years, before 1630 and after up to 1640, resident in London, near Tower Hill, in pursuit of his calling as teacher of mathematics. During the civil war he seems to have resided in Bermuda, where he had a government grant as schoolmaster and where in 1662 he conducted another survey. He was in England in 1667, probably on a visit. He died in Bermuda in 1675 (Henry Wilkinson, The Adventures of Bermuda [Oxford: Humphrey Milford, 1933]).
20. Trigonometrie (4th ed., London: R. and W. Leybourn, 1661), extract from "To the Reader."

> But all that hath been done these many hundred years is not comparable to that which hath been effected in our times, by the Honourable Lord John Nepair, Baron of Merchiston: who by an invention of Logarithms, takes away those difficulties that were in the practice thereof. Which invention hath been illustrated and much perfected by the labors of M. Henry Brigs. Neither is M. Edward Wright[21] to be forgotten though his endeavors were soonest prevented.

The Flemish mathematician, Simon Stevin,[22] ranks among the most original scientific thinkers of the second half of the sixteenth century. To Simon Stevin is given the credit of the invention of decimal fractions. The trigonometry of Stevin was on a par with that of his contemporaries. In his work of 1608[23] he reduced the trigonometric formulas relative to right-angled triangles to six and stated explicitly that these six formulas were sufficient to solve all problems.

Pitiscus and his contemporaries must yet be considered. Their work is of special importance to this study, because it was through this group of mathematicians, some of whom have already been mentioned, that trigonometry finally acquired complete emancipation from astronomy.

2. FRANCISCO MAUROLYCO

Francisco Maurolyco (1494-1575), a Greek, lived in Italy and was an Abbot in Messina. Though far removed from the center of Italian culture, his place among Italy's foremost men of this period is not an isolated one. Maurolyco imbibed the spirit of the ancients and adopted their methods; he imitated his immediate predecessors and contemporaries and either reproduced or extended their work.[24]

Maurolyco was the first to make a Latin translation of the Sphaerics of Menelaus from the Arabic and Hebrew manuscripts; this appeared in 1558.[25] In the translation he followed the practice of the Arabs in replacing the chord by the sine.[26] In 1558[27] he also made a translation of the works of Theodosius and Autolycus pertaining to spherical geometry and produced his own treatise of two books on the sphere. The work of Maurolyco shows the influence of the trigonometry of Regiomontanus and also that of Geber; the knowledge of the latter he obtained through Peter Apian's edition of Gerard

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21. Norwood was probably referring to Edward Wright's translation of Napier's work which was published posthumously in 1618.

    Edward Wright, mathematician and hydrographer, was born at Garveston, Norfolk, about 1558. When and in what circumstances Wright turned his attention to nautical matters is doubtful. All his nautical writings relate to navigation considered as a branch of mathematics; it is on these that his fame rests. Wright died in London in 1615 (James Wilson, History of Navigation, prefixed to J. Robertson, The Elements of Navigation 4th ed., London: J. Nourse, 1780).

22. Simon Stevin was born at Bruges in 1548, and was active for a time in Antwerp as cashier and bookkeeper. In 1583 he matriculated as a student in the University of Leyden where later on he taught mathematics, Prince Maurice of Nausau being one of his pupils. In 1600, he organized the mathematical teaching at Leyden University, the teaching being in the national language as opposed to the University's instructions in Latin. Stevin died in 1620, presumably at the Hague (George Sarton, "Simon Stevin of Bruges," Isis, 21:241-303, 1934).

23. This is Wiskonstige Gedachtenissen which Willebrod Snellius translated into Latin in the same year and gave the title, Hypomnemata mathematica (A. von Braunmühl, op. cit., pp. 226-227). See p. 107.

24. Frederico Napoli, "Alla vita ed ai lavori di Francisco Maurolico," Bulletin di Bibliografia e di Storia delle Scienze Mathematische e Fisiche publicato da B. Boncompani, 9:1-22, Rome, 1876.

25. A. von Braunmühl, op. cit., p. 14.

26. Ibid., p. 16.

27. This edition was not consulted in this study. The same treatises are found in the 1626 edition of Elementorum Euclids libri XVI cum additionibus Maurolyci et Candallae et Opticorum libri IV, edited by Mersenne. Of special interest in this edition is a trigonometric summary entitled "Ad Lectorem" which precedes the work of Theodosius. In all probability it was written by Mersenne. The author states that the definitions and propositions have been taken from Euclid, Ptolemy, Regiomontanus, Copernicus, and others.

of Cremona's translation.[28] His notations for the trigonometric functions, namely, "sinus 2^m arcus" and "sinus 1^m arcus" can be traced to the work of Peter Apian[29] wherein are found the expressions "sinus rectus secundus" and "sinus rectus primus."

Since Mersenne's edition of the treatise of Maurolyco contains no demonstrations or figures, what is deemed necessary here for the sake of clarity will be added.

In Book I of his treatise on the sphere, Maurolyco gives the definitions of "sinus rectus," "sinus totus," "sinus secundus," and "sinus versus." It should be recalled that these were the only functions that appeared in the De triangulis libri quinque of Regiomontanus.

The first proposition of Book I is a statement regarding the foregoing definitions:

> Lines from any point on the periphery of a semicircle perpendicular to the diameter are right sines of the arcs from that point to the diameter. The portions of the diameter from the perpendicular to the periphery of the circle are the vers sines of the arcs. Those parts from the center to the perpendicular are the "sinus secundi."

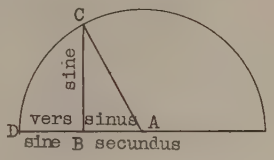

Fig. 71

The other forty-eight propositions of this book deal directly or indirectly with a right spherical triangle. Proposition 9, for instance, implies the Sine Law for a spherical right triangle.

"If in a triangle formed from the arcs of great circles on the surface of a sphere, the 'sinus totus' is to the sine of an acute angle as the sine of the arc subtending another angle is to the sine of the arc opposite the acute angle, that angle will be a right angle."

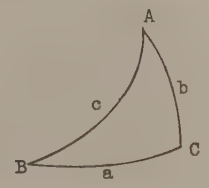

Fig. 72

$$\frac{\text{"sinus totus"}}{\text{sine A}} = \frac{\sin c}{\sin a}$$

The more familiar formula is:

$$\sin a = \sin c \ \sin A.$$

Maurolyco imitated Regiomontanus in his expression "casus" for the portions of the base which are produced when a perpendicular is drawn to it from the opposite vertex.[30] This is exemplified by Proposition 33:

> Si duo circuli maiores in superficie sphaerae angulum actum contineant, & in uno eorum signentur duo puncta, à quibus arcus circulorum maiorum perpendiculares ad reliquum ducantur: Ratio sinus arcus inter casus perpendicularium ad sinum arcus inter puncta signata cadentis componetur ex duabus: quarum una est ratio sinus totius ad sinum secundum unius arcus perpendicularium: altera est ratio sinus secundi acuto angulo

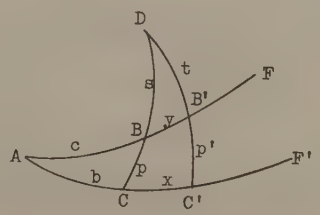

Fig. 73

28. Instrumentum primi mobilis... Nuremberg, 1543. (Not at U. M.).
29. See p. 37.
30. See p. 26.

praedicto debiti ad sinum secundum reliqui arcus perpendicularis.

Let AF and AF' be the two great circles of a sphere which form the acute angle A. Let B and B' be the two points on the one circle from which the perpendiculars p and p' are drawn meeting AF' at C and C' respectively. The theorem of Maurolyco states that the following equation is true:

$$\frac{\sin x}{\sin y} = \frac{\text{"sinus totus"} \times \cos A}{\cos p \times \cos p'}$$

To verify this theorem it is necessary to prolong p and p' until they intersect at D. Triangle CDC' has two right angles, hence the sides opposite these angles are quadrants. From this it follows that sin s = cos p, sin t = cos p', and sin x = sin D.

Consider triangle DBB' and apply the sine law:

(1) sin D/sin y = sin DBB'/sin t = sin ABC/cos p'.

Since triangle ABC is a right triangle,

(2) "sinus totus" cos A/cos p = sin ABC. (This is Proposition 21.)

By substituting for sin D, sin t and sin ABC in (1), the required equation is produced:

(3) sin x/sin y x "sinus totus" cos A/cos p x cos p'.

Proposition 6 of Book II is the statement of the theorem of Ptolemy[31] relative to a cyclic quadrilateral:

Si quadrilaterum rectilineum circulo inscriptum fuerit: Quod sub duabus eius diametris continetur rectangulum aequale est duobus ijs, quae sub oppositis lateribus comprehenduntur, coniunctim sumptis rectangulis.

In Proposition 10 Maurolyco states:

In a right triangle formed from three great circles on the surface of a sphere, the 'sinus' of the sum of the two arcs including an acute angle to the 'sinus' of their difference is as the sum of the 'sinus totus' and 'sinus secundus' of this same acute angle to the difference of the same quantities.[32]

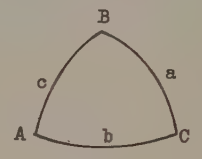

Fig. 74

$$\frac{\sin (c + b)}{\sin (c - b)} = \frac{1 + \cos A}{1 - \cos A}$$

In Proposition 15 Maurolyco establishes the identity:

$$\frac{1 + \cos A}{1 - \cos A} = \frac{\cos^2 \frac{A}{2}}{\sin^2 \frac{A}{2}}$$

In Proposition 16 he combines Propositions 10 and 15:

31. See p. 5 and p. 32.

32. Maurolyco has "ad differentiam eorundem sinuum" which translated literally is "to the difference of the same sines," meaning the 'sinus totus' and the 'sinus secundus.'

Item, sinus aggregati arcuum acutum angulum comprehendentium ad sinum differentiae eorundem arcuum est sicut quadratum, quod ex sinu secunde dimidij anguli acuti ad quadratum, quod ex sinu eiusdem dimidij.

The identity he wishes to establish is:

$$\frac{\sin (c + b)}{\sin (c - b)} = \frac{\cos^2 \frac{A}{2}}{\sin^2 \frac{A}{2}}$$

Maurolyco considers the tangent and cotangent functions, even though he does not define them with the other trigonometric functions in Book I. He uses them under the names "umbra versa" and "umbra recta" respectively. Thus in Proposition 30 he states:

> The sine of any arc to the 'sinus secundus' is as the gnomon[33] to the 'umbra recta' of the same arc. Also the 'sinus secundus' to the sine of this arc is as the gnomon to the 'umbra versa' of the arc.

This is no more than saying sin A/cos A = 1/cot A and cos A/sin A = 1/tan A.

The tangent function is also employed by Maurolyco in Proposition 32:

In a right-angled triangle formed from arcs of the great circles on the surface of a sphere, the 'sinus totus' to the sine of an arc subtending an acute angle is as the 'umbra versa' of the other acute angle to the 'umbra versa' of the side, not yet considered, of this same angle.

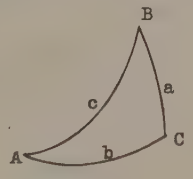

Fig. 75

$$\frac{\text{'sinus totus'}}{\sin a} = \frac{\tan B}{\tan b}$$

The more familiar formula is:

$$\sin a = \tan b \cot B.$$

In this treatise of two books on the sphere, Maurolyco establishes trigonometric relations between sides and angles of spherical triangles, but he does not apply these formulas to the formal solution of spherical triangles.

In Part III of "Scritti inediti di Francesci Maurolico," Federico Napoli[34] has given "Maurolyci siculi Geometricarum Quaestionum." Napoli mentions that the original manuscript has the date "XII Mar. M.D.L.V." for Book I and "XXIX Mar. M.D.L.V." for Book II. The trigonometric portion of Book I begins with Question 33: "Circuli diameter est pedum .75.chorda cuiusdam arcus est .72. quanta erit chorda residui de semicirculo?"

Quadratum .72. est .5184. quadratum uero diametri .5625. ab hoc illud subtrahe: et relinquentur .441. huis radix 21. chorda residui di semicirculo.

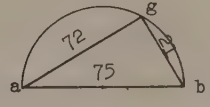

Fig. 76

Questions 34 to 38 and their accompanying figures are similar to the propositions and figures of the Tractatus of Peurbach.[35] Question 38 is the first question

33. In astronomy the gnomon is an instrument or apparatus for measuring the altitudes, declinations, etc., of the sun and stars. The gnomon is usually a pillar or column. Its use in Proposition 30 is comparable to the "sinus totus" of the other propositions.
34. Op. cit., pp. 50-113
35. See pp. 31-32.

relative to the measuring of angles: "Quomodo mensurantur anguli triangulorum, quorum cognita sunt latera?"

Here Maurolyco establishes his definitions for "sinus rectus," "sinus secundus" and "sinus versus."

Consider triangle abc, having a right angle at c; bc is the 'sinus rectus' of angle a; ca is the 'sinus secundus' of this same angle; and the 'sinus versus' is cd. On the other hand ac is the 'sinus rectus' of angle abc; bc is the 'sinus secundus'; the 'sinus versus' is the excess of ab over bc.

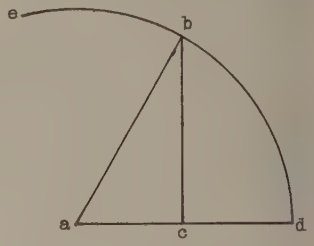

Fig. 77

Question 39 inquires about the ratio of the sides of a triangle when the angles are known.

Spherical trigonometry begins with Question 49: "If two sides which include an acute angle in a spherical right triangle are known, in what manner can I find the third side?"

Before answering this question, Maurolyco constructs a single figure. This figure, namely 78, becomes for Maurolyco fundamental in his solution of right triangles.

The general construction of the figure will be discussed first; after that its application to Question 49 will be considered.

Construct two quadrants ab and cb intersecting at a right angle in b. Draw the quadrant ae and then construct the quadrant cd intersecting at f so that

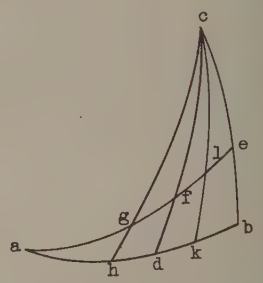

Fig. 78

(1) $1/\sin cf = \sin cf/\sin ce$;

hence (2) $1/\cos fd = \cos fd/\cos a$.

In right triangle adf,

(3) $\cos fd = \cos af/\cos ad$,

so (4) $\cos ad/\cos af = \cos af/\cos ad \cos a$;

hence (5) $\cos a = \cos^2 af/\cos^2 ad$.

But (6) $\cos a = \sin ad \cos af/\sin af \cos ad$,

hence (7) $\cos^2 af/\cos^2 ad = \sin ad \cos af/\sin af \cos ad$,

so (8) $\cos af/\cos ad = \sin ad/\sin af$, where $af = ad$.

Hence (8) is true if and only if $af + ad = 90^\circ$.

Since $af + fe$ is a quadrant, $ad = fe$; $ad + db$ is a quadrant, so $af = db$.

Now construct two other quadrants cgh and clk, either one of which may be arbitrary, so that

(9) $\sin cg/\sin cf = \sin cf/\sin cl$.

In right triangle agh

(10) $\cos ag/\cos ah = \cos gh = \sin cg$.

In right triangle afd

(11) $\cos af/\cos ad = \cos fd = \sin cf$.

In right triangle akl,

(12) $\cos al/\cos ak = \cos kl = \sin cl$.

Hence (9) becomes

(13) cos ag cos ad/cos ah cos af = cos af cos ak/cos ad cos al;

which can also be expressed as

(14) cos ag $\cos^2$ ad/cos ah $\cos^2$ af = cos ak/cos al.

Apply (5) to (14) and obtain

(15) cos a = cos ag cos al/cos ah cos ak.

But in triangle agh,

(16) cos a = sin ah cos ag/sin ag cos ah;

so (15) and (16) produce

(17) sin ah cos ag/cos ah sin ag = cos ag cos al/cos ah cos ak;

whence (18) sin ah/sin ag = cos al/cos ak = sin le/sin kb;

using triangle akl and making similar substitutions it can be ascertained that

(19) cos ah/cos ag = cos le/cos kb, where ah = ag and le = kb

(18) and (19) are true if and only if ah = le and ag = kb. From this it easily follows that gf = dk and fl = hd.

The application of the sine law gives

l/sin a = sin ae/sin ab = sin al/sin lh = sin af/sin fd = sin ag/sin gh.

With the given data of Question 49, Fig. 78 can be constructed in the following manner: Construct the two quadrants ab and cd intersecting in a right angle at b. Since GH and ah are known, the following relation can be used in order to determine be:

$$\frac{\sin (ga + ah)}{\sin (ga - ah)} = \frac{1 + \cos a}{1 - \cos a} = \frac{1 + \sin ce}{1 - \sin ce} .$$

After be is determined, ed with the point f can also be determined. The three points a, f, and e determine the quadrant ae. On ae apply the known arc ag; do the same to ah on ab, the points h, g, and f determine the quadrant hc. These are all the lines of Fig. 78 that are necessary for the construction of triangle agh. In order to answer the question proposed, Maurolyco makes use of triangle agh where ag and ah, which include angle a, are known. He supposes arc ag to be 30° and ah to be $32^\circ 42'$. He wants to use the "sinus secundus" of these arcs which he does indirectly by considering the "sinus rectus" of ge and hb and gives the values according to a table with the radius equal to 10000. He applies the formula sec gh = cos ag cos ah and gives a numerical answer to the proposed problem.

Maurolyco proposes other questions pertaining to Fig. 78. For instance, Question 50 asks how to find ah when ag and gh are known, while Question 51 is concerned about the arc ag when gh and ha are known.

In Question 63, which pertains to the construction of the midpoint of an arc, Maurolyco states this identity:

$$\frac{\text{"sinus totus"} + \sin ce}{\text{"sinus totus"} - \sin ce} = \frac{\sin^2 cm}{\sin^2 bm} = \frac{\sin (ga + ah)}{\sin (ga - ah)} ;$$

this is followed by "Quae omnia sphaericis Menelai ac nostris abunde sunt demonstrata." Recalling how Fig. 78 was constructed, it can be verified that the above formula is equivalent to:

$$\frac{1 + \cos a}{1 - \cos a} = \frac{\cos^2 a/2}{\sin^2 a/2} = \frac{\sin (ga + ah)}{\sin (ga - ah)} .$$

which is identical with the formula Maurolyco gave in his treatise on the sphere, which has already been considered.[36]

In Question 64, Maurolyco discusses the secant function. He calls the table of secants "tabula benefica" and says it is not less useful than the "tabula fecunda" of Regiomontanus.[37]

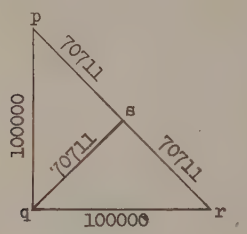

Fig. 79(a)

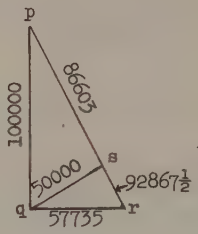

Fig. 79(b)

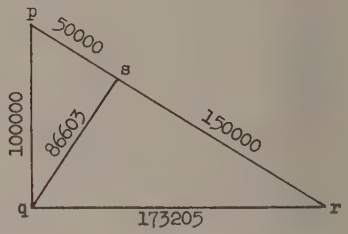

Fig. 79(c)

Maurolyco considers a plane triangle pqr, in which q is a right angle. He draws qs perpendicular to pr, making triangles pqr, psq, and qsr similar, hence corresponding sides are proportional. Maurolyco says that in order to know everything concerning these triangles it is necessary to use the table of sines, the "tabula fecunda" and the "tabula benefica." If angle p is found in the table of sines, the line qs is given; in the "tabula fecunda," the line qr is given; in the "tabula benefica," the line pr is given. Maurolyco is here imitating Rheticus in his triangles of first and second series;[38] Maurolyco does not discuss the co-functions.

The question now arises whether the "tabula benefica" to which Maurolyco refers,[39] was produced independently by him or whether it was merely a reproduction of the table of secants of Rheticus. In his Geometria rotundi of 1583, Thomas Finck[40] remarks that Maurolyco has reproduced the tables of Rheticus with a few changes; he has only added a new name to them.[41]

In the preface to his Primum mobile,[42] Magini speaks in the defence of Maurolyco. While not denying the credit of the priority of the introduction of the use of the secant or hypotenuse to Rheticus,[43] he claims that although the tables of Rheticus were published in 1551 and those of Maurolyco in 1558, the work of the latter was done independently of the Canon doctrinae triangulorum of Rheticus.

In the Opuscula mathematica published at Venice in 1575 a treatise with the title "De sphaera liber unus," bearing the date February 17, 1569, contains the definitions of the trigonometric functions that Maurolyco used in his treatise on the sphere. Since the definitions of "umbra recta" and "umbra versa" were not given in this treatise, it might be well to quote them from the "De sphaera liber unus."[44]

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36. See p. 68.
37. "Addam his duabus tabellum [sine and tangent] nostram beneficam non minus utilem faecunda."
38. See p. 56.
39. These tables were written in 1555 and published in 1558.
40. Thomas Finck was born in Flemsburg in 1561. He visited the Universities of Jena, Wittenberg, and Heidelberg; he died at Copenhagen in 1656 (A. von Braunmuhl, op. cit., p. 187). See pp. 88-90.
41. Ibid., p. 151.
42. See p. 99.
43. "...[Rheticus] ipsi primus Secantium siue Hypotenusarum usum introduxit."
44. Page 13.

Cotangent is the shadow cast by the gnomon when it is perpendicular to the horizontal plane; it is zero when the sum is at its zenith; it is infinity when the sun is at the horizon. The tangent is the shadow cast by the gnomon when it is perpendicular to the vertical plane...

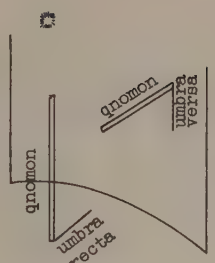

Fig. 80

3. FRANCIS VIETA

It has already been stated that Vieta was the foremost mathematician of France in the sixteenth century.[45] The trigonometrical notions contained in his various works show the influence of Rheticus. To portray the advances made in trigonometry by Vieta, it will be necessary to analyze that portion of the Canon mathematicus seu ad triangula cum appendicibus published in 1579 at Paris by Francis a Schooten.

The Canon mathematicus seu ad triangula cum appendicibus is composed of four parts, only the first two of which were published in 1579 - the "Canon mathematicus," which contains the table of trigonometric lines with some accessory tables and the "Universalium inspectionum ad Canonem mathematicum liber singularis," which gives the formulas for the solution of plane and spherical triangles with a great number of numerical results; the two other parts, devoted to astronomy, have not been published.[47]

Vieta was certainly cognizant of the work of Rheticus, because he adopted for his "Canon mathematicus" the triangles of three series of Rheticus. He established their relations with an inscribed and circumscribed circle but changed the nomenclature of Rheticus after he had used it for the first few pages; Vieta's terminology and abbreviations for the trigonometric functions in this work are:[48]

Series I	Sine of acute angle	Sinus A	(sine)
	Sine of the rest	Sinus $\bar{A}$	(cosine)
	"Sinus totus"		
Series II	Fecunda of the acute angle	Fecunda A	(tangent)
	"Hypotenusa Fecundae"	Hypotenusa A	(secant)
	"Sinus hypotheticus"		
Series III	Fecunda of the rest	Fecunda $\bar{A}$	(cotangent)
	Hypotenuse of the rest	Hypotenuse $\bar{A}$	(cosecant)
	"Sinus hypotheticus"	Total	(radius)

The "Canon mathematicus" with its appendices has six tables, the first table of which gives minute by minute the values of the six trigonometric lines as is indicated by a portion of it[49] given on the next page. For the construction of this table Vieta applied the method given by Ptolemy in his Almagest which was ameliorated by the Arabs and adopted in the Occident chiefly by Peurbach and Regiomontanus who made use of it to compute their tables of sines, that is, they took for their point of departure an arc of 15° called a "kardaga."[50]

The second table is the "Canonion triangulorum laterum rationalium," established on the proposition: "If in a right-angled triangle having h for the hypotenuse, b for the base and p for the perpendicular and the semi-difference $(h - p)/2 = 1$, then we have $h = b^2/4 + 1$, and $p = b^2/4 - 1$; if we give b the successive values of an arithmetic

45. See p. 64.
46. Only secondary sources were available, viz., Frédéric Ritter, op. cit., pp. 37-48 and Karl Hunrath, loc. cit.
47. Frédéric Ritter, op. cit., p. 37.
48. Ibid., p. 40.
49. Karl Hunrath, op. cit., p. 218.
50. See p. 33.

Trianguli Plani Rectanguli

Quadrans circuli XC part. Angulus rectus, Hypotenusae congruus	Circumferentia Perpendiculo congrua	Hypotenusa 100,000 Perpendiculum	Basis	Basis 100,000 Perpendiculum	Hypotenusa	Perpendiculum 100,000 Basis	Hypotenusa	Io ad dati Residua Basi congrua
		E canone Sinuum Prima		E canone Faecunda Faecundissimoque Serie Secunda		E canone Faecunda Faecundissimoque Tertia		
	Part IIII Scrup.							
	XXX	7,846	99,691 7	7,870	100,309 2	1,270,621	1,274,550	XXX
		29	23	29	23	4,708	4,695	
	XXXI	7,875	99,689 4	7,899	100,311 5	1,265,913	1,269,855	XXIX
		29	22		23	4,674	4,659	
	XXXII	7,904	99,687 2	7,929	100,318 8	1,261,239	1,265,196	XXVIII
		29	25	29	25	3,842	3,827	
	LX	8,716	99,619 5	8,749	100,382 0	1,143,006	1,147,372	I

progression; in the table of values of h and p thus formed, the differences will again be constant."[51]

Vieta formed a series of 1422 triangles, called primitive triangles, in which h and p are rational; he made successively in each primitive triangle one of the sides equal to 100000 or "sinus totus"; he obtained a table divided into three series, corresponding to the series given in the "Canon" and giving the exact values of two of the six trigonometric lines.[52]

The third table, "Ad logisticem per Ἑζεχονταδας tabellae," is a table of multiplication in the form of a right-angled triangle which gives immediately in degrees and minutes, the product n x n'/60 for all the numbers n and n' included between 0 and 60.[53]

"Fractionum apud mathematicos usitandarum alterius in alterum reductionibus tabella adcommodata" is the fourth table and gives for the circle for the Egyptian year, for the day and for the hour and for their principal subdivisions, the quotients of the one by the other and also by the most commonly used integers.[54]

The fifth table, called "Mathematici canonis epitome" gives the values of the trigonometric lines from degree to degree and the length of the arc expressed in parts of the radius.[55]

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51. Frédéric Ritter, op. cit., p. 46.
52. Ibid., p. 47.
53. Idem.
54. Idem.
55. Ibid., pp. 47-48.

The sixth table or "Canon triangulorum ad singulas partes quadranti circuli secundum Εξεχονταδων logisticem" gives the value of the six trigonometric lines from degree to degree, the radius 1 being divided into 60 parts each part into 60 primes and each prime into 60 seconds; thus, sin 45° = 42 25' 35"[58]

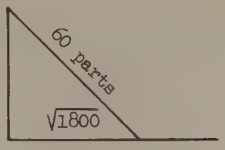


Fig. 81

1800 parts = 42.4264 parts.
.4264 parts = 25.5840 primes.
.5840 primes = 35.04 seconds.

Having made known the construction of the Canon of triangles for rational sides, Vieta passes to the trigonometry in his "Universalium inspectionum ad Canonum mathematicum liber singularis." He employs the same letters in the triangles to designate the same elements; in the right triangle ABC, AB is the hypotenus; CB, the perpendicular opposite the acute angle A; the base opposite angle B, remainder or complement of angle A.[57]

He gives as functions of the radius the values of the sides of the inscribed polygons of 3, 4, 6, 10, and 15 sides and the relation that exists between these trigonometric lines which permits an easy calculation of the tables; he employs formulas equivalent to:[58]

$$\cos^2 A = R^2 - \sin^2 A; \qquad (R-\cos A)R = 2\sin^2 \tfrac{1}{2} A;$$

$$4\sin^2 \tfrac{1}{2}(A - B) = (\sin A - \sin B)^2 + (\cos B - \sin A)^2;$$

$$\sin(60^\circ + A) = \sin A - \sin(60^\circ - A);$$

$$\cot \tfrac{1}{2} A = \sec A + \cot A; \qquad \cot A = \cot 2A + \csc 2A;$$

$$\tan \tfrac{1}{2} A = \cot A - \csc A; \qquad \tan A = \cot 2A - \csc 2A;$$

$$\tan(45^\circ + A/2) = 2\tan A + \tan(45^\circ - A/2);$$

$$\sec A - \tfrac{1}{2}\tan(45^\circ - A/2) + \tfrac{1}{2}\tan(45^\circ - A/2).$$

The formulas he employs for the solution of right triangles are:[59]

$$a = \frac{b \sin A}{R}; \qquad c = \frac{a \sec B}{R}; \qquad b = \frac{a \tan B}{R};$$

$$b = \frac{a \sin B}{R}; \qquad c = \frac{b \sec A}{R} \qquad a = \frac{b \tan A}{R}.$$

For the solution of oblique triangles, Vieta uses the following formulas:[60]

$$\sin A - \sin B = \frac{2\cos \tfrac{1}{2}(A + B)\sin \tfrac{1}{2}(A - B)}{R};$$

$$\cos A - \cos B = \frac{2\sin \tfrac{1}{2}(A + B)\sin \tfrac{1}{2}(A - B)}{R};$$

$$\cot \tfrac{1}{2}(A + B) = \frac{\sin A - \sin B}{\cos A - \cos B}.$$

56. Idem.
57. Idem.
58. Ibid., p. 49.
59. Ibid., p. 50.
60. Ibid., pp. 50-61.

In his solution of oblique triangles, Vieta resolves all the cases, except where three sides are given, by proportionality of sides to the sines of the angles opposite the sides; for the case of three sides, he follows the others in decomposing the triangle into right triangles and using the formula:

$$BC \times CD - \tfrac{1}{2}(\overline{AD}^2 - \overline{BD}^2 - \overline{AB}^2).$$ [61]

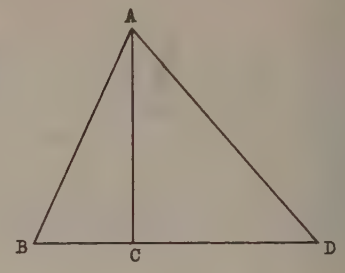


Fig. 82

For spherical triangles he employs the same notation as for plane triangles; he establishes the fact that a spherical right triangle can be calculated by means of the total sine and two other elements; some of his formulas are:[62]

$$\sin a = \sin c \sin A; \qquad \tan A = \tan a \csc B;$$
$$\sin a = \tan b \cot B; \qquad \tan A = \sec c \cot B;$$
$$\sin a = \sin c \frac{\cos B}{\cos b}; \qquad \tan A = \tan c \frac{\sec b}{\sin B}$$
$$\sin a = \sec b \frac{\cos B}{\csc C}; \qquad \tan A = \tan c \frac{\cos B}{\sin b}.$$

In spherical oblique triangles, Vieta follows the ancients and Regiomontanus in decomposing the triangle into two right triangles by an arc of a great circle perpendicular to one of the sides and passing through the vertex of the angle opposite; he uses the tangent:[63]

$$\frac{\tan \frac{1}{2}(A + B)}{\tan \frac{1}{2}(A - B)} = \frac{\sin A + \sin B}{\sin A - \sin B}.$$ [64]

The Opera mathematica of Vieta comprises sixteen treatises, some of which, either wholly or in part, are devoted to the doctrine of triangles. The trigonometric portions of these treatises will now be considered.

In "Ad logisticem speciosam notae priores," which was edited in Paris by Guillaume Baudry in 1631, there are some interesting propositions which combine algebra with geometry. For instance, Proposition 46: "From two right-angled triangles construct a third right-angled triangle."

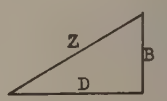


Fig. 83(a)

Vieta lets the hypotenuse of the third triangle equal the product of the hypotenuse of the first and second, then he determines the remaining sides of the third in the following manner:

$$Z^2 = D^2 + B^2$$
$$X^2 = G^2 + F^2$$
$$Z^2X^2 = G^2D^2 + G^2B^2 + F^2D^2 + B^2F^2$$
$$Z^2X^2 = (B\,G + D\,F)^2 + (G\,D - B\,F)^2 \quad \text{or}$$
$$(B\,G - D\,F)^2 + (G\,D + B\,F)^2$$

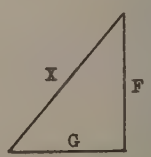


Fig. 83(b)

61. Ibid., p. 51.
62. Ibid., p. 52.
63. Ibid., p. 53.
64. A. von Braunmühl, op. cit., p. 169, says that F. Ritter is not correct in his statement that this formula is given in Canon mathematicus... of 1579; it is given for the first time by Vieta in the "Variorum de rebus mathematicis responsorum liber VIII" of the Opera mathematica, i.e., in 1593. See p. 85.

Therefore the base and perpendicular may be either B C $\pm$ D F or G D $\mp$ B F respectively, giving rise to two cases of a third triangle.

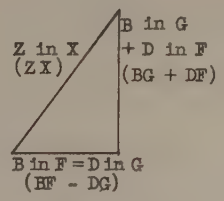


Fig. 83(c_1)

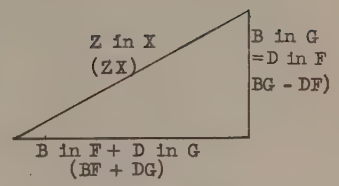


Fig. 83(c_2)

In Proposition 48, Vieta considers two equiangular and equilateral right triangles: "A duobus triangulis rectangulis aequalibus & aequiangulis, tertium triangulum rectangulum, constituere."

By applying the first case of Proposition 46 to the two congruent triangles, the third triangle can be easily determined.

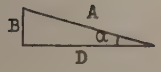


Fig. 84(a)

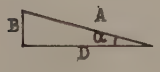


Fig. 84(b)

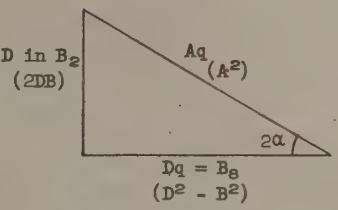


Fig. 84(c)

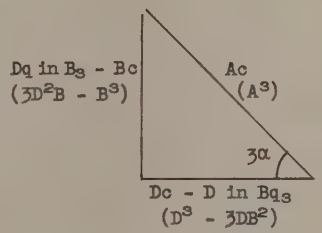


Fig. 85

Vieta calls angle α of triangle ABD a simple angle, and angle 2α of the third triangle, a double angle.

In Proposition 49, Vieta constructs a triangle with a triple angle by applying Proposition 46 to the right triangles with simple and double angles respectively. He obtains the triangle in Fig. 85.

In Proposition 50 he obtains a triangle with a quadruple angle by applying Proposition 46 to triangles with simple and triple angles respectively.

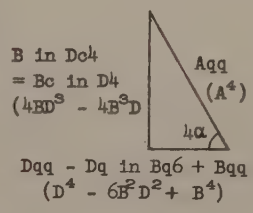


Fig. 86

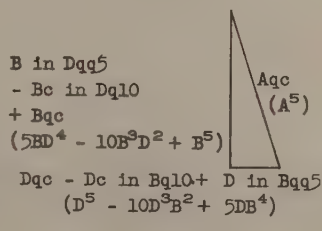


Fig. 87

Vieta gives one more triangle, namely, one with a quintuple angle and then concludes that any triangle derived from an original one in such a way that the power of the length of the hypotenuse is equal to the multiplicity of the angle, then the formulas for the perpendicular and base of an angle of multiplicity m become:

$$\text{Perpendicular} = [m\ BD^{m-1} \mid \frac{m(m-1)(m-2)B^3D^{m-3}}{3!} \cdots]$$

$$\text{Base} = [D^m - \frac{m(m-1)}{2!} B^2 D^{m-2} + \frac{m(m-1)(m-2)(m-3)}{4!} B D^{m-4} \cdots]$$

By multiplying both members of the above equations by the hypotenuse, the general formulas for the sine m and cosine m are produced. Thus Vieta has established the following formulas:[65]

65. See p. 80 for another reference to these theorems.

$$\sin \alpha = B/A$$

$$\sin 2\alpha = 2\,BD/A^2$$

$$\sin 3\alpha = (3\,BD^2 - B^3)/A^3$$

.....

$$\sin m\alpha = (m\,BD^{m-1} - \frac{m(m-1)(m-2)}{3!}B^3D^{m-3}\ldots)/A^m$$

$$\cos \alpha = D/A$$

$$\cos 2\alpha = (D^2 - B^2)A^2$$

$$\cos 3\alpha = (D^3 - 3\,DB^2)A^3$$

.....

$$\cos m\alpha = (D^m - \frac{m(m-1)}{2!}B^2D^{m-2} + \frac{m(m-1)(m-2)(m-3)}{4!}B^4D^{m-4}\ldots)A^m$$

The treatise, "De aequationum recognitione et emendatione tractatus duo," edited by Jean Laquenay in 1615, contains the remarkable theorem of Vieta pertaining to the irreducible case of a cubic equation; it is Theorem III of Chapter VI:

If $A^3 - 3\,AB^2 = B^2D$, where $B > D/2$, and $3\,B^2E - E^3 = B^2D$, then there are two right triangles each with hypotenuse equal to B, such that the acute angle subtended by the perpendicular of the first is three times the acute angle subtended by the perpendicular of the second. Twice the base of the first is D and twice the base of the second is A. E will be the base of the second augmented or dimished by the length of the square root of three times the perpendicular of the second.

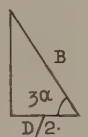


Fig. 88(a)

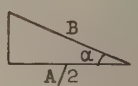


Fig. 88(b)

By observing Fig. 88(a) and Fig. 88(b) it can easily be seen that $A/B = 2\cos\alpha$ and $D/B = 2\cos 3\alpha$; hence the first equation in Theorem III becomes: $(2\cos\alpha)^3 - 3(2\cos\alpha) = 2\cos 3\alpha$.

The problem Vieta proposes in order to demonstrate the use of this theorem is:

"1 C - 300 N aequetur 432" and "300 N - 1C aequetur 432." He says there are two triangles each having the hypotenuse 10, the acute angle which is subtended by the perpendicular of the first is three times as great as the acute angle subtended by the perpendicular of the second, but twice the base of the first is 432/100. Therefore the unknown is equal to twice the base of the second and is also equal to the base of the second plus or minus the square root of three times the perpendicular of the second.

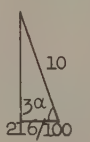


Fig. 89(a)

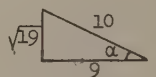


Fig. 89(b)

The equations are:

$$x^3 - 300\,x = 432 \quad \text{and}$$

$$300\,x - x^3 = 432.$$

$B = 10$, $D = 432/100$, hence

$$\cos 3\alpha = 216/1000.$$

Hence 3α is known, so α is known and cos α is also known. Vieta finds cos α to be 9/10, hence $A/2 = 9$, and the perpendicular of the second triangle is $\sqrt{19}$. Therefore the three roots of the proposed equations are: $18,\ 9 \pm \sqrt{57}$.

The trigonometric solution of the cubic equation occurs twice in the treatise, "Supplementum geometriae," which was printed at Tours in 1593 by Mettayer, namely, in Propositions 16 and 17:

If two isosceles triangles have the equal sides of the one equal to the equal sides of the other, but the base angles of the second are three times the base angles of the first, then the cube of the base of the first minus three times the product of the base of the first and the square of one of the equal sides equals the product of the base of the second and the square of one of the equal sides; i.e.:

$$\overline{AC}^3 - 3\,\overline{CD}^2 \times AC = CE \times \overline{CD}^2.$$

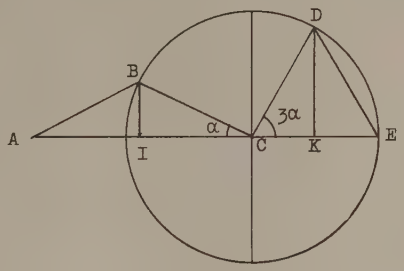


Fig. 90

Si duo triangula fuerint aequicrura singula, & ipsa alterum alteri cruribus aequalia, angulus autem, quem is qui est ad basin secundi relinquit è duobus rectis, sit triplus anguli qui est ad basin primi: solidum triplum sub base primi & cruris communis quadrato, minus cubo è base primi, aequale est solido sub base secundi & cruris communis quadrato.

The treatise "Theoremata ad sectiones angulares," as the title appears in Opera mathematica, was not published by Vieta but was printed at Paris in 1615 under the title Ad angularium sectionum analyticen theoremata a Fr. Vieta excogitata et demonstrationibus confirmata ab Al. Andersonio. This treatise deals, in part, with some general formulas of chords, sines, cosines and tangents of multiple arcs in terms of the trigonometric lines of the simple arcs. For example, in Theorem I, Vieta remarks:

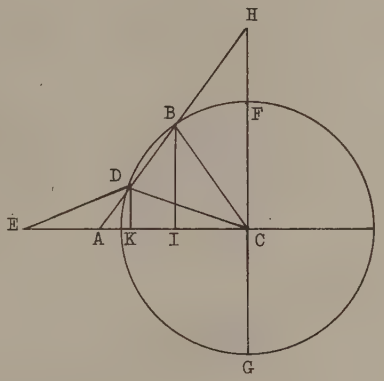


Fig. 91

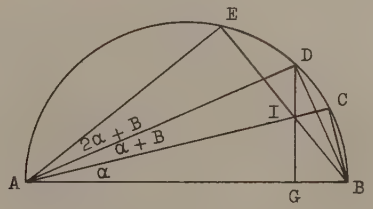


Fig. 92

If there are three right-angled triangles of which the acute angle of the first differs from the acute angle of the second by the magnitude of the acute angle of the third, the sides of the three triangles have this relation:

Angle EAB = $2\alpha + \beta$
Angle DAB = $\alpha + \beta$
Angle CAB = α

The perpendicular of the third equals the perpendicular of the first multiplied by the base of the second minus the product of the perpendicular of the second and the base of the first.

$$\sin\alpha = \sin(2\alpha + \beta)\cos(\alpha + \beta) - \sin(\alpha + \beta)\cos(2\alpha + \beta)$$

The base of the third equals the product of the bases of the first and second plus the product of their perpendiculars.

$$\cos\alpha = \cos(2\alpha + \beta)\cos(\alpha + \beta) + \sin(\alpha + \beta)\sin(2\alpha + \beta).$$

In Theorem II, Vieta establishes the formulas:

$$\sin(2\alpha + \beta) = \sin(\alpha + \beta)\cos\alpha + \cos(\alpha + \beta)\sin\alpha$$

$$\cos(2\alpha + \beta) = \cos(\alpha + \beta)\cos\alpha - \sin(\alpha + \beta)\sin\alpha$$

Theorems III and IV are statements equivalent to the formulas:

$$\frac{\sin\alpha}{\sin 2\alpha} = \frac{\sin(n-1)\alpha}{\sin(n-2)\alpha - \sin n\alpha}$$

$$\tfrac{1}{2} : \cos\alpha = \cos(n-1)\alpha : [\cos(n-2)\alpha + \cos n\alpha]$$

In Theorem VI Vieta considers the equations for multiple angles:

$2\cos\alpha = AD = x$

$2\cos 2\alpha = AE = x^2 - 2$

$2\cos 3\alpha = AF = x^3 - 3x$

$2\cos 4\alpha = AG = x^4 - R\,x^2 + 2$

$2\cos 5\alpha = AH = x^5 - 5x^3 + 5x$

$2\cos 6\alpha = AI = x^6 - 6x^4 + 9x^2 - 2$

. . . . . .

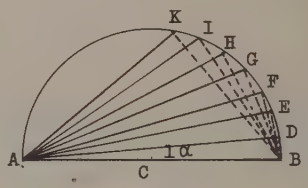


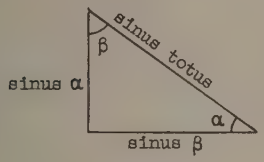
Fig. 93

At the end of Theorem VI, Vieta presents this table for determining the coefficients; the first column in the table comprises the coefficients of the second terms of the above equations excluding the first equation; the second column refers to the coefficients of the third terms, etc. The words "negative" and "affirmative" refer to the alternate signs.

Primae negatae	Secundae affirmatae	Tertia negatae	Quartae affirmatae	Quintae negatae	Sextae affirmatae	...	Nonae negatae
2							
3							
4	2						
5	5						
6	9	2					
7	14	7					
8	20	16	2				
9	27	30	9				
10	35	50	25	2			
11	44	77	55	11			
12	54	112	105	36	2		
13	65	156	182	91	13	...	
14	77	210	294	196	49		
15	90	275	450	318	140	...	
16	104	352	660	672	336		
17	119	442	935	1122	714	...	
18	135	546	1287	1782	1386		2
19	152	665	1729	2717	2508	...	19
20	170	800	2275	4604	4290		100
21	189	952	2940	5733	7007	...	385

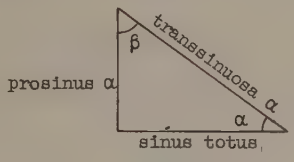
The "Variorum de rebus mathematicis responsorum liber VIII" was edited at Tours by Mettayer in 1593; it is an extract of the various questions from Sept livres de questions mathematiques which Vieta proposed to Scaliger.[66] The trigonometric portion of this treatise begins with Chapter XIX and deals with right and oblique plane and spherical triangles.

In the first proposition Vieta considers the triangles of the three series, the concept being that of Rheticus but the nomenclature his own.



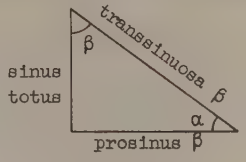
First Series

Fig. 94(a)



Second Series

Fig. 94(b)



Third Series

Fig. 94(c)

To these three triangles, Vieta adds three more and calls them "mixed," i.e., they are obtained from an interchange of elements of the three foregoing series.

Perpendiculum fiet simile differentiae inter transsinuosam anguli acuti & sinum reliqui è recto. Basis, sinui acuti. Hypotenusa, prosinui ejusdem.

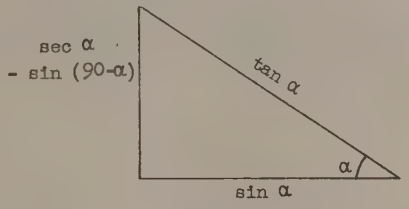


Fig. 95

Perpendiculum fiet simile differentiae inter sinum anguli acuti & transsinuosam reliqui è recto. Basis, sinui reliqui è recto. Hypotenusa, prosinus ejusdem.

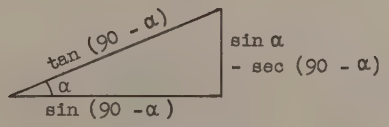


Fig. 96

Perpendiculum fiet simile transinuosae anguli acuti. Basis, transsinuosae reliqui è recto. Hypotenusa, adgregato prosinus acuti & prosinus reliqui e recto.

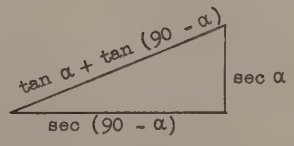


Fig. 97

66. Joseph Scaliger (1540-1603) published several pieces of Greek and Latin verse, in which he affirmed that he had discovered a rigorous solution of the quadrature of a circle, the construction of two mean proportionals, the duplication of the cube and the trisection of an angle (Frédéric Ritter, op. cit., p. 24).

For the right triangle he established such formulas as:

$$\frac{\text{hypotenuse}}{\text{sinus totus}} = \frac{\text{perpendicular}}{\text{sine of acute angle}} \cdot \frac{\text{base}}{\text{cosine}}$$

$$\frac{\text{base}}{\text{sinus totus}} = \frac{\text{hypotenuse}}{\text{secant of acute angle}}$$

$$\frac{\text{perpendicular}}{\text{sinus totus}} = \frac{\text{hypotenuse}}{\text{cosecant of acute angle}}$$

} which determine the angle when the hypotenuse and perpendicular or base are given.

For the oblique plane triangle Vieta establishes formulas similar to:

$$\frac{2\,bc}{b^2 + c^2 - a^2} \quad \frac{1}{\cos A}$$

$$\frac{c + b}{c - b} = \frac{\tan \frac{C + B}{2}}{\tan \frac{C - B}{2}}$$

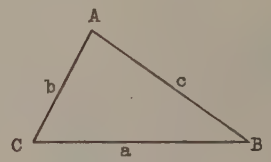


Fig. 98

Spherical trigonometry begins with Theorem VIII; this theorem pertains to the solution of a right spherical triangle.

Dato latere, quod angulo recto opponitur, ac uno ex obliquis angulo, dantur latera reliqua, ac angulus reliquus.

Given the side which subtends the right angle and one of the oblique angles, find the remaining sides and angle.

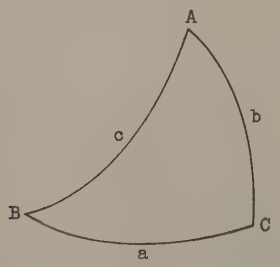


Fig. 99

Ut sinus totus ad sinum lateris dati, ita sinus anguli dati ad sinum lateris angulo dato oppositi.

$\sin a = \sin c \sin A.$

In a similar manner Vieta establishes all the formulas for the right spherical triangle that he had given in the Canon mathematicus of 1579.[67]

In regard to the polar triangle and Vieta's use of it, Braunmühl says:[68]

> It is, as far as it is known to me, a hitherto undetermined question, whether Vieta knew and employed the real polar or supplementary triangle of a spherical triangle...

---

67. See p. 75.
68. "Zur Geschichte der sphärischen Polardreieckes," Bibliotheca Mathematica, 3:65, Berlin, 1898.

Quoting other authors, Braunmühl says that Cantor[69] maintains that Vieta is the first one to mention the polar triangle, but that his definition, namely, "Si sub apicibus singulis propositi tripleuri sphaerici, describantur maximi circuli: tripleurum ita descriptum, tripleuri primum propositi, lateribus et angulis est reciprocum," is not clear. Delambre's opinion is quite the same as that of Cantor:[70]

> Si Viète avait eu l'idée de ce triangle supplémentaire, aurait-il négligé d'en expliquer les propriétés et les facilités qu'il offre pour les démonstrationes de certaines theorems.

Ritter[71] claims Vieta had the knowledge of the polar triangle, but that his demonstrations are faulty.

In his later work, Braunmühl[72] gives the assurance of the fact that Vieta's reciprocal figure is quite the same as the polar triangle. He arrives at this conclusion because the theorems of Vieta are arranged in such a manner that a succeeding theorem is the dual of the one immediately preceding it.[73] Had Vieta given the derivation of his formulas, the questioning of his knowledge of the theory of polars would never have occurred.

Tropfke[74] claims that the brief treatment given to polar triangles by Vieta was so ambiguous that his contemporaries failed to comprehend it, hence they ascribed the priority of its use in Europe to the Hollander, Willibrord Snellius (1581-1626).

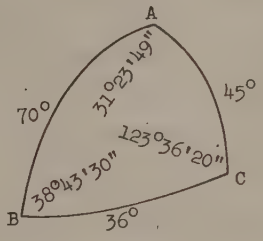


Fig. 100

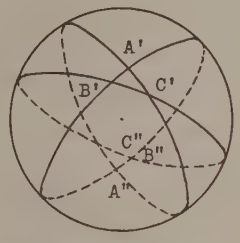


Fig. 101

The spherical triangle which Vieta employs in one of his demonstrations is similar to Fig. 100. To obtain the reciprocal triangle Vieta constructs A' and A", the poles of BC; B' and B", the poles of AC; and C' and C", poles of AB. Thus eight triangles are formed as shown in Fig. 101 (Vieta's treatise does not have this figure).

The eight triangles, which are in reality four pairs of congruent triangles, are:

A'B'C' congruent to A"B"C"
A"B"C' congruent to A'B'C"
A"B'C" congruent to A'B"C'
A"B'C' congruent to A'B"C"

Except for the dotted lines Fig. 102 resembles the figure of Vieta; it is a conventional representation of Fig. 101.

The Roman numerals refer to the following table found in the treatise of Braunmühl:[75] (In this treatise Braunmühl corrects the error of Vieta in ascribing 90° to the angles at C in place of the correct value of 110°.)

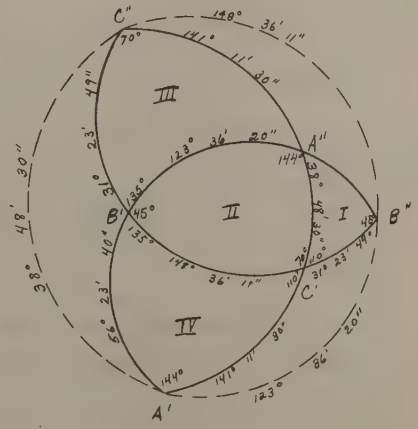


Fig. 102

69. Vorlesungen über Geschichte der Mathematik (Leipzig: B. G. Teubner, 1882), vol. II, p. 556.
70. Jean Delambre, Histoire de l'astronomie du moyen âge (Paris: Courcier, 1819), p. 478.
71. Op. cit., p. 56.
72. Vorlesungen über Geschichte der Trigonometrie (Leipzig: Teubner, 1900), vol. I, p. 183.
73. See p. 84.
74. Geschichte der Elementar-Mathematik (2nd ed., Berlin: De Gruyter and Co., 1923), vol. V, p. 125.
75. "Zur Geschichte des sphärischen Polardreieckes," Bibliotheca Mathematica, 3:70, Berlin, 1898.

$$\text{I.}\quad \left.\begin{array}{ll} a'' = A, & A'' = a, \\ b'' = B, & B'' = b, \\ c' = 180^\circ - C, & C' = 180^\circ - c, \end{array}\right\} \quad \text{for } \Delta\, A''B''C',$$

$$\text{II.}\quad \left.\begin{array}{ll} a'' = 180^\circ - A, & A'' = 180^\circ - a, \\ b' = B, & B' = b, \\ c' = C, & C' = c, \end{array}\right\} \quad \text{for } \Delta\, A''B'C',$$

$$\text{III.}\quad \left.\begin{array}{ll} a'' = A, & A'' = a, \\ b' = 180^\circ - B, & B' = 180^\circ - b, \\ c'' = C, & C'' = c, \end{array}\right\} \quad \text{for } \Delta\, A''B'C'',$$

$$\text{IV.}\quad \left.\begin{array}{ll} a' = 180^\circ - A, & A' = 180^\circ - a, \\ b' = 180^\circ - B, & B' = 180^\circ - b, \\ c' = 180^\circ - C, & C' = 180^\circ - c, \end{array}\right\} \quad \text{for } \Delta\, A'B'C'.$$

Vieta's demonstration of the polar triangle was somewhat modified by his immediate successors, namely, Lansberg,[76] Pitiscus,[77] and Simon Stevin.[78]

The following are the theorems for which Vieta obtains the formulas for the polar triangle. The parallel arrangement is not given by Vieta.

XV. Three sides being given determine the angles.

$$\frac{\sin a \sin b}{\cos c \mp \cos a \cos b} = \frac{1}{\cos C}.$$[79]

If $a = b$, then

$$\frac{1}{\sin a} = \frac{\csc c/2}{\csc C/2}.$$

XVI. Three angles being given determine the sides.

$$\frac{\sin A \sin B}{\cos A \cos B \pm \cos C} = \frac{1}{\cos c}$$

If $A = B$, then

$$\frac{1}{\sin A} = \frac{\sec C/2}{\sec c/2}.$$

XVII. Two sides and the included angle being given, the remaining angles are also given.

Suppose b, c, and A are given; determine C.

$$\frac{\sin A \csc b}{\cot c \mp \cos A \cot b} = \frac{1}{\cot C}.$$

If $b = c$, then

$$\frac{1}{\sec c} = \frac{\cot A/2}{\cot C}$$

XVIII. Two angles and the included side being given, the remaining sides are also given.

Suppose B, C, and a are given; determine c.

$$\frac{\sin a \csc B}{\cot C \pm \cos a \cot B} = \frac{1}{\cot c}$$

If $B = C$, then

$$\frac{1}{\sec C} = \frac{\tan a/2}{\cot c}$$

XIX. If two sides and the included angle are given, the remaining side will also be given.

Suppose b, c, and A are given; determine a.

$$\frac{\csc b \cos c}{\cos A \mp \cot b \cot c} = \frac{1}{\cos a}$$

XX. If two angles and the included side are given, the remaining angle will also be given.

Suppose B, C, and a are given; determine A.

$$\frac{\csc B \csc C}{\cos a \pm \cot B \cot C} = \frac{1}{\cos A}$$

XXI. Given two sides and an angle opposite one of them, the angle opposite the other side is also given.

XXII. Given two angles and a side opposite one of them, the side opposite the other angle is also given.

76. See p. 97.
77. See p. 103.
78. See p. 108.
79. This is similar to the formula of Regiomontanus, except that Regiomontanus uses the vers sine; see p. 31.

For these two theorems Vieta states the Sine Law:

Sinus laterum sunt similes sinibus angulorum quibus latera opponuntur.

Vieta concludes this section of his treatise by a lengthy list of trigonometric identities, some of which are:

Ut sinus peripheriae ad sinum totum, ita sinus totus ad transsinuosam complementi. $\sin a = 1/\sec a.$

Ut sinus complementi primae ad sinum complementi secundae, ita transsinuosa secundae ad transsinuosam primae. $\cos a/\cos b = \sec b/\sec a.$

Ut rectangulum sub prosinu peripheriae primae & sinu secundae ad rectangulum sub prosinu secundae & sinu primae, ita transsinuosa primae ad transsinuosam secundae. $\dfrac{\tan a \sin b}{\tan b \sin a} = \dfrac{\sec a}{\sec b}.$

Given the sum of two arcs, the ratio of whose secants are known, to find the arcs.

$$\frac{\sec a}{\sec b} = \frac{\csc (a + b)}{\tan b + \cot (a + b)} \cdot [(a + b) < 90^\circ]$$

If the sum of two arcs are given and also the ratio of the secants of the two arcs, find the arcs.

$$\frac{\sec a}{\sec b} = \frac{\csc (a + b)}{\tan b - \cot (a + b)} \cdot [(a + b) > 90^\circ\ a < 90^\circ,\ b < 90^\circ]$$

Among the final formulas of this treatise, the following statement occurs:

Ut adgregatum similium sinuum ad differentiam eorundem, ita prosinus dimidiae summae peripheriarum ad prosinum dimidiae differentiae earundem, vel ita prosinus complementi dimidiae differentiae peripheriarum ad prosinum complementi dimidiae summae.

$$\frac{\sin \alpha + \sin \beta}{\sin \alpha - \sin \beta} = \frac{\tan \frac{(\alpha + \beta)}{2}}{\tan \frac{(\alpha - \beta)}{2}} = \frac{\cot \frac{(\alpha - \beta)}{2}}{\cot \frac{(\alpha + \beta)}{2}}$$

In the perusal of the trigonometric works of Vieta, very evident conclusions are reached. Vieta aided pre-logarithmic computation by his extension of the trigonometric tables of Rheticus. His principal advance in trigonometry was the systematic method he employed in the application of algebra. In both plane and spherical trigonometry he worked freely with all six of the trigonometric functions. He obtained many of the fundamental identities of plane trigonometry algebraically, notably the trigonometric functions of multiple angles.

His innovations in trigonometry are somewhat comparable with his invention of the literal symbolism. This he introduced into his trigonometric work of 1579 in connection with a general unknown angle, and this was "the initial and crucial step toward a new mathematics."[80]

Vieta was not only the greatest mathematician of his time, he led his European contemporaries and was imitated by his immediate successors.

80. L. C. Karpinski, "Descartes and the Modern World," Science, 89:151, February 17, 1939.

4. MAURICE BRESSIEU

The Metrices astronomicae of Maurice Bressieu[81] published in Paris in 1581, two years after the appearance of Vieta's Canon mathematicus, shows that Bressieu was familiar with the work of Rheticus. Three of the four books into which this treatise is divided pertain to trigonometry. Book II, having the title "De sinibus, adscriptis et hypotenusis" contains the following definitions.

The sine of an arc is the perpendicular from one end of the arc to the diameter of the circle which terminates at the other end of the arc.

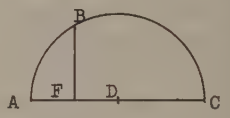


Fig. 103

When at the extremity of the diameter a perpendicular is erected, the perpendicular is called the "adscripta" of the arc included between the two semidiameters. The radius prolonged to meet the "adscripta" is the hypotenuse of the same arc.

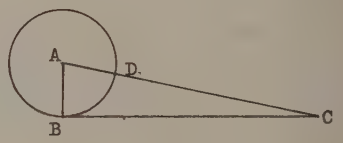


Fig. 104

The complement of an angle is the difference between itself and a right angle, whether the angle be larger or smaller than a right angle.

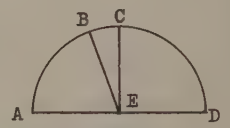


Fig. 105

The first six propositions of Book II with their accompanying figures are similar to those in the Tractatus of George Peurbach.[82] Tables of sines, tangents, and secants make up the remainder of Book II. The tables are constructed in a manner similar to the sixth table of Vieta in Canon mathematicus.[83] The following is an excerpt from the table of sines:

grad	42			47			43			44			
m. arcuum	Sinus			Sinus			Sinus			Sinus			m. arcuum
	g̃	m̃	Z̃	g̃	m̃	Z̃	g̃	m̃	Z̃	g̃	m̃	Z̃	
0	40	8	52	44	35	19	40	55	12	41	40	46	60
1		9	39		34	37		55	57		41	31	59
2		10	25		33	55		56	43		42	16	58
59		54	26	43	53	35	41	40	1	42	24	10	1
60											25	35	0

The complementary function can be read from this table also.

81. The date of the birth of Bressieu is unknown; he died in 1608. He was professor of the "College de France" in Paris from 1576 to 1608.

82. See pp. 31-32.

83. See p. 74.

NICOLAI COPERNICI TORINENSIS DE REVOLVTIONIBVS ORBIVM coelestium, Libri VI.

Habes in hoc opere iam recens nato, & ædito, studiose lector, Motus stellarum, tam fixarum, quàm erraticarum, cum ex ueteribus, tum etiam ex recentibus obseruationibus restitutos: & nouis insuper ac admirabilibus hypothesibus ornatos. Habes etiam Tabulas expeditissimas, ex quibus eosdem ad quoduis tempus quàm facilli me calculare poteris. Igitur eme, lege, fruere.

Ἀγαθοῦ τύχη δόξα.

Norimbergæ apud Ioh. Petreium, Anno M. D. XLIII.

Canon subtensarum in circulo rectarum linearum.

Circūferentiæ.	Semisses dupl. circūferen.	Differentiæ.	Circūferentiæ.	Semisses dupl. circūferen.	Differentiæ.
gr. ser			gr. ser		
0 10	291	291	6 10	10742	289
0 20	582		20	11031	
0 30	873		30	11320	
0 40	1163		40	11609	
0 50	1454		50	11898	
1 0 1	1745		7 0	12187	
1 10	2036		10	12476	
1 20	2327		20	12764	
1 30	2617		30	13053	288
1 40	2908		40	13341	
1 50	3199		50	13629	
2 0 2	3490		8 0	13917	
2 10	3781		10	14205	
2 20	4071		20	14493	
2 30	4362		30	14781	
2 40	4653	291	40	15069	
2 50	4943	290	50	15356	287
3 0 3	5234		9 0	15643	
3 10	5524	290	10	15931	
3 20	5814		20	16218	
3 30	6105		30	16505	
3 40	6395		40	16792	
3 50	6685		50	17078	
4 0 4	6975		10 0	17365	
4 10	7265		10	17651	286
4 20	7555		20	17937	
4 30	7845		30	18223	
4 40	8135		40	18509	
4 50	8425		50	18795	
5 0 5	8715		11 0	19081	
5 10	9005		10	19366	285
5 20	9295		20	19652	
5 30	9585		30	19937	
5 40	9874	290	40	20222	
5 50	10164	289	50	20507	
6 0	10453	289	12 0	20791	

Canon subtensarum in circulo rectarum linearum.

Circūferentiæ.	Semiss. subtend dup. cir.	Differentiæ.	Circūferentiæ.	Semisses subtend. dup. cir.	Differentiæ.
gr. sect			gr. sec.		
10	21076	284	10	31178	276
20	12350		20	454	6
30	21644		30	730	6
40	21928		40	32006	6
50	22212		50	282	5
13 0	22495	283	19 0	557	5
10	22778		10	832	5
20	23062		20	33106	5
30	23344		30	381	4
40	23627		40	655	4
50	23900	282	50	929	4
14 0	24192		20 0	34202	4
10	24474		10	315	3
20	24750		20	748	3
30	25038	281	30	35021	3
40	25319		40	293	2
50	25601		50	562	2
15 0	25882		21 0	832	2
10	26163		10	36108	1
20	26443	280	20	379	1
30	26724		30	650	1
40	27004		40	920	0
50	27284		50	37190	0
16 0	27564	279	22 0	460	270
10	27843		10	739	269
20	28122		20	999	9
30	28401		30	38268	9
40	28680		40	538	8
50	28959	278	50	805	8
17 0	29237		23 0	39073	8
10	29515		10	39341	7
20	29793		20	39608	7
30	30071	277	30	875	7
40	30348		40	40141	6
50	30625		50	408	6
18 0	30902		24 0	674	266

Plate 10. Nicolas Copernicus. De revolutionibus orbium coelestium, Libri VI, Nuremberg, 1543.

MAVRICII BRESSII

GRATIANOPOLITANI REGII

ET RAMEI MATHEMATICARVM

LVTETIÆ PROFESSORIS METRICES

Astronomicæ libri quatuor.

HÆC MAXIMAM PARTEM NOVA EST RERVM
Astronomicarum & Geographicarum per plana sphericáque triangula
dimensionis ratio, veterique impendiò expeditior
& compendiosior.

AD POMPONIVM BELLEVREVM SACRI CONSISTORII CONSILIARIVM, REGISQVE LEGATVM.

PARISIIS,

Apud Ægidium Gorbinum, sub insigne Spei, è regione gymnasij Cameracensis.

M. D. LXXXI.

CVM PRIVILEGIO REGIS.

qualium AB trianguli latus, tanquam circuli radius est 1 sex.ī. talium dabitur BC ex canone adscriptarum, & AC ex canone hypotenusarum. Ergo qualium AB per hypothesim datur partium, talium BC & AC dabuntur.

Exemplum. Sit angulus A 36 g̃.52 m̃.12 ẗ. AB 8. Erit igitur angulus C 53 g̃.7 m̃.48 ẗ. Qualium ergo AB est 1 sex.ī, talium BC erit 45 ex canone adscriptarum. AC 1 sexag.ī.15 g̃. ex canone hypotenusarum. Ergo qualium AB 8, talium BC erit 6. AC 10. Nec vero nos fugiat, posse vtranque πτῶσιν huius propositionis, tam ex finibus, quàm ex adscriptis & hypotenusis expediri.

¶ Quod si propositum triangulum sit obliquangulum, tamen quia dati sunt anguli, quorum sinus ex thuius sunt in ratione laterum iis oppositorum, lateribus assignabimus numeros sinuum angulorum iis oppositorum. Hoc pacto data ratione laterum trianguli, & vno ex ipsis dato, reliqua dabuntur in iisdem numeris, per 19 lib. 7 elem.

PROPOS. III.

Duobus trianguli rectanguli lateribus datis, angulos tertiumque latus inuenire.

PTOLEMAICA METHODVS.

REpetito primo antecedentis theorematis diagrammate, dentur trianguli ABC duo latera, & primo latus AC, cum vno eorum quæ circa rectum, nempè CB. Oportet omnes angulos, latusque tertium inuenire. Cum igitur AC & BC datæ sint ex hypothesi, qualium AC vt semidiameter circuli ABC est 120, talium dabitur BC. Per canonem vero chordarum vel sinuum, arcus BEC: totidémque partium & angulus BAC, qualium 2 recti sunt 360. Ergo qualium 4 recti sunt 360, talium dabitur angulus BAC, reliquusque C. Dato autem arcu BEC, dabitur residuus ad semicirculum BDA, rectáque AB ex canone chordarum vel sinuum, talium dabitur partium qualium AC 120. Qualium igitur AC datur partium ex hypothesi, itémque BC, talium dabitur & AB. Potest etiam exquiri latus AB, etiam non cognitis angulis, ad hunc modum. Quoniam quadratum ex AC æquale est duobus quadratis ex AB, BC. Subducto quadrato rectæ BC, ex quadrato rectæ AC, relinquetur quadratũ rectæ AB. Vnde recta AB cognoscetur talium partium qualium AC & BC. Similis inuestigationis lateris AB apud Ptolemæum exemplum est cum alibi, tum lib. 6 μεγ. συνταξ. capite quod inscribitur προταθεν κανονιον ἐπιπεδικῶν.

Exemplum. Sit AC 10. BC 6. Qualium igitur AC vt dimetiens circuli ABC est 120, talium BC erit 1 sex.ī. 12 g̃. tanquam arcus BEC chorda. Ipse autem arcus BEC ex can. chordarum vel sinuum 73 g̃. 44 m̃. 24 ẗ. totidemque partium & angulus BAC, qualiũ vtique 2 recti sunt 360, ergo qualium 4 recti sunt 360 partium, talium angulus BAC erit 36 g̃. 52 m̃. 12 ẗ. Reliquus vero C talium 53 g̃. 7 ī. 48 ẗ. Quoniam vero arcus BEC est 73 g̃. 44 ī. 24 ẗ. Residuus ad semicirculum BDA erit 106 g̃. 15 m̃. 36 ẗ. qualium tota circuli triangulum ABC circunscribentis peripheria est 360. Eius autem chorda AB 96. qualium vtique AC 2 sex.ī. Ergo qualium AC est 10, talium AB erit 8. Aut etiam quia AC est 10, ipsius quadratum erit 100. Quadratum rectæ BC 36. Quo deducto ex 100, restat quadratum rectæ AB 64. Cuius latus AB erit 8.

RECENTIOR METHODVS.

Repetatur secundum antecedentis theorematis diagramma. Cum igitur AC & BC in iisdem datæ sint partibus: Qualium AC, vt radius, est 1 sex.ī, talium dabitur BC per 19 lib. 7 elem. & ex can. sinuum angulus BAC, ex indéque rectũ reliquus C, & arcus AD, eiusque sinus AB ex can. sinuum. Ergo qualium AC ex hypothesi data est partium, talium dabitur AB. Exemplum. Sit AC 10. BC 6. Qualium igitur AC est 1 sex.ī, talium BC erit 36. Et complemẽtum arcus DA, siue angulus BAC erit 36 g̃. 52 m̃. 12 ẗ. Reliquus ergo AD, angulusue C, 53 g̃. 7 ī. 48 ẗ. eiusque sinus AB ex can. sinuũ 48, qualium nimirum AC est 1 sex.ī. Qualium ergo AC 10, talium AB erit 8.

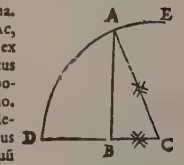


Plate 11. Maurice Bressius. Astronomicae libri quatuor, 1581.

The table of tangents and secants with their co-functions are arranged in the following manner:

m. arcuum	grad 42				42				44				44				m. arcuum
	Adscriptae				Hypotenusa				Adscriptae				Hypotenusa				
	sex	g̃	m̃	2̃	sex	g̃	m̃	2̃	sex	g̃	m̃	2̃	sex	g̃	m̃	2̃	
0		54	1	27	1	20	44	17		57	56	28	1	23	24	35	60
1			3	21			45	33			58	30			26	0	59
2			5	15			46	48		58	0	32			27	25	58

The numbers in the secant columns are in red.

Book III deals with plane triangles. The first proposition is the Sine Law for which Bressieu gives three proofs: "Trianguli latera ad sinus angulorum iis oppositorum easdem habent rationes."

In his first proof, Bressieu divides the given triangle in the usual way into two right triangles. By assuming the Sine Law for a right triangle he obtaines two proportions; their quotient produces the Sine Law for the given triangle. In this proposition Bressieu states that the radius is equal to the sine of a right angle.

The second proof is true if, and only if, the circumscribing circle is a unit circle; Bressieu neglects to mention this condition and gives the proof as follows:

> Circumscribe the triangle ABC and construct the perpendiculars DE, DF, and DG; draw the lines DA, DB and DC. Then AB, BC, and CA are bisected at the points E, G, and F. Angles BDA, ADC, and BDC are bisected by the perpendiculars DE, DF, and DG, hence angle BDA is double angle EDA, so ADE and ACB are equal. Similarly ABC is equal to ADF, and GDF is equal to BAC. Also, AB equals 2 AE, CA equals 2 AF; hence BA/AE equals CA/AF equals CB/BG. But AE is sin ADE or ACB; AF is sin ADF or ABC; BG is sin BDG or BAC. Therefore the sine law is proved.

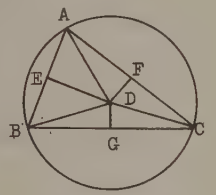


Fig. 106

In the third proof Bressieu constructs a figure similar to Fig. 107; by describing the circle with the side BC of the given triangle as radius; by prolonging BA and constructing the perpendicular CH and the parallel EF, Bressieu is able to prove the Sine Law by means of similar triangles.

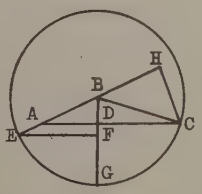


Fig. 107

In most of the propositions Bressieu gives several proofs, heading them, as the case might be, with "Ptolemaica Methodus," "Recentior Methodus," or "Nostro Methodus."

An innovation, possibly original with Bressieu,[84] is that of marking the given parts of a figure in the solution of triangles. For instance, Proposition 3: "Given two sides of a right triangle, determine the angles and the remaining side."

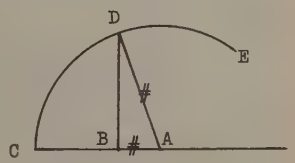


Fig. 108

84. This is contrary to a statement of Cajori: "Perhaps the earliest author to use strokes [to indicate the given parts of a triangle] was Philipp van Lansberge.... He published a trigonometry...the dedicatory epistle of which bears the date 1591" [Florian Cajori, A History of Mathematical Notations (Chicago: The Open Court Publishing Co., 1929), vol. II, p. 148]. Bressieu's trigonometry antedated that of Phillip van Lansberg.

The remaining propositions of Book III deal with the solution of right and spherical triangles.

The thirty-eight propositions of Book IV pertain to spherical trigonometry. The first nine propositions consider the geometry of the sphere and show a great similarity to Book III of the De triangulis libri quinque of Regiomontanus.

Except for terminology, Proposition 17 is almost identical with a proposition in the Treatise on the Quadrilateral of Nasir ad-Dīn:[85]

> I say that the ratio of the sine of the arc CA to the sine of the arc AE is composed of the ratio of the sine of the arc CD, to the sine of the arc DF and the ratio of the sine of the arc FB to the sine of the arc BE.

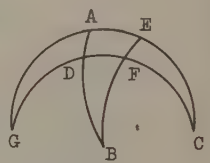


Fig. 109

The proof is similar to the one given by Naṣir ad-Dīn.

Bressieu makes mention of the errors made by Copernicus in his trigonometry;[86] he attributes these mistakes also to Regiomontanus, but the De triangulis libri quinque does not have these errors, as has been pointed out.[87]

Although Bressieu made no exceptional advance in trigonometry and appears to have borrowed much from his predecessors, there is a remarkable independence in the underlying thought in some of his proofs.

5. THOMAS FINCK

The Geometria rotundi libri XIIII of Thomas Finck, published at Basle in 1583,[88] had a certain degree of influence on the trigonometry of this period. Credit is given to Finck for introducing the words "tangent" and "secant" into trigonometry and for giving the first proof for the Tangent Theorem.[89]

> Recta sinibus connexa est tangens peripheriae, aut eam secans. Proposuimus ut in circulo inscriptas, sic in semicirculo sinus perpendendos, sed & rectas sinibus connexas. Eas plenioris intellectus causa in tangentes & secantes dislocamus: verbis si hac in re, non nova, novis, tamen, ut speramus, accommodatis.

Finck remarks that the lines cutting the circumference of a circle or tangent to it are related to the sine, hence, he uses the names tangent and secant for these lines, and hopes these terms will be used in this trigonometric connection.

Vieta did not approve of these terms, because of the confusion that might arise with these same terms in geometry. His disapproval was overlooked and subsequent writers adopted the terminology of Finck instead of the "prosinus" and "transinuosa" of Vieta; however, Finck's expression "sinus secundus" for the vers sine found no acceptance.[90]

Finck did not define the trigonometric functions as parts of a right triangle as Rheticus and Vieta had done, but considered the trigonometric lines as lines of a circle.[91]

85. See p. 11.
86. See pp. 49, 51, 52.
87. See p. 27, prop. 45; p. 30, prop. 25.
88. Because this book was not available the following secondary sources have been used: A. von Braunmühl, Vorlesungen uber Geschichte der Trigonometrie (Leipzig: Teubner, 1900), vol. I; Johann Tropfke, op. cit., p. 35; Florian Cajori, op. cit.
89. J. Tropfke, op. cit., p. 35.
90. F. Cajori, op. cit., p. 150.
91. A. von Braunmühl, op. cit., p. 187.

grad. 78			78			79			79			80			80 arcuū.		
	Adscriptæ			Hypotenusæ			Adscriptæ			Hypotenusæ			Adscriptæ			Hypotenusæ			
arcuū.	sex.	g.	m.	s.	sex.	g.	m.	s.	sex.	g.	m.	s.	sex.	g.	m.	s.	sex.	g.	m.	s.	
0	4	42	16	40	4	48	35	2	5	8	40	14	5	14	26	51	5	40	16	37	60
1			40	54			58	46		9	9	4			55	37			51	23	59
2		43	5	12		49	22	31			38	3		15	23	37		41	26	14	58
3			29	38			46	26		10	7	6			52	12		42	1	17	57
4			54	8		50	10	23			36	17		16	20	48			36	24	56
5		44	18	35			34	23		11	5	23			49	34		43	11	35	55
6			43	11			58	22			34	38		17	18	5			46	55	54
7		45	7	54		51	22	33		12	4	1			46	56		44	22	26	53
8			32	36			46	44			33	24		18	15	48			17	53	52
9			57	27		52	11	0		13	2	55			44	48		45	33	45	51
10		46	22	13			35	17			32	28		19	13	49		46	9	32	50
11			47	12			59	44		14	2	9			43	0			45	39	49
12		47	12	10		53	24	11			31	57		20	12	14		47	21	59	48
13			37	14			48	46		15	1	44			41	34			37	40	47
14		48	2	24		54	13	21			31	39		21	10	53		48	54	4	46
15			27	39			38	6		16	1	31			40	23		49	10	37	45
16			52	57		55	2	51			31	43		22	9	54			47	1	44
17		49	18	22			27	43		17	1	52			39	39		50	23	42	43
18			43	46			52	36			32	17		23	9	24		51	0	53	42
19		50	9	17		56	17	36		18	2	46			39	21			37	37	41
20			34	49			42	36			33	15		24	9	19		52	24	46	40
21		51	0	28		57	7	43		19	4	6			39	30			52	8	39
22			26	8			32	51			34	43		25	9	43		53	29	17	38
23			52	0			58	9		20	5	28			40	0		54	6	50	37
24		52	17	48		58	23	28			36	21		26	10	18			44	28	36
25			43	46			48	58		21	7	20			40	50		55	22	23	35
26		53	9	53		59	14	29			38	31		27	11	25		56	0	12	44
27			35	53			40	3		22	9	36			42	4			37	53	33
28		54	2	4	5	0	5	37			40	53		28	12	44		57	16	7	32
29			28	17			31	17		23	12	18			43	38			54	20	31
30			54	31			57	0			43	45		29	14	32		58	32	40	30
31		55	20	58		1	22	57		24	15	20			45	39		59	11	1	29
32			47	29			48	55			47	3		30	16	47			49	46	28
33		56	13	59		2	14	55		25	18	54			48	8	6	0	28	36	27
34			40	36			40	58			50	49		31	19	30		1.	7	31	26
35		57	7	15		3	7	8		26	22	40			50	57			46	43	25
36			34	1			33	20			54	47		32	22	24		2	25	58	24
37		58	0	48			39	38		27	26	38			54	8		3	5	1	23
38			27	42		4	25	57			59	19		33	25	53			44	25	22
39			54	43			52	27		28	31	54			57	43		4	23	32	21
40		59	21	45		5	18	57		29	4.	3		34	29	34		5	4	4	20
41			48	58			45	41			36	38		35	1	41			43	43	19
42	5	0	16	12		6	12	21		30	9	21			33	48		6	23	39	18
43			43	31			39	10			42	18		36	6	15		7	3	58	17
44			1	10	54		7	6	0		31	15	20			38	43			54	26	16
45			38	23			53	0			48	17		37	10	26		8	24	47	15
46			2	6	0		8	0	1		32	21	30			43	50		9	5	23	14
47			33	32			27	8				54	43		38	26	33			46	2	13
48	5	3	1	17			54	25		33	28	0			49	17		10	26	56	12
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50			56	52			48	47			35	0			55	24			49	27	10
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53	5	21	1			51	11	26		36	36	18			35	0			53	34	7
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55		6	17	36		32	26	55		37	24	18			41	58		15	17	37	5
56			46	2			34	46			58	33		43	25	36			59	35	4
57	7	14	19			53	2	44		38	55	3			49	30		44	23	24	3
58			43	2			30	43		39	7	24			57	0		17	24	6	2
59	5	8	21	27			58	46			41	41			57	0		18	6	51	1
grad.	18				11				10				10				9				0 complem.

grad. 18			71			19			70			20			69 arcuū.		
	Sinus.			Sinus.			Sinus.			Sinus.			Sinus.			Sinus.		
arcuū.	g.	m.	s.	g.	m.	s.	g.	m.	s.	g.	m.	s.	g.	m.	s.	g.	m.	s.	
0	18	32	28	57	3	48	19	32	3	56	43	52	20	31	16	56	22	54	60
1		33	27		3	29		33	2		43	31		32	15		22	32	59
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3		35	27		2	50		35	1		42	50		34	13		21	49	57
4		36	27		2	30		36	0		42	30		35	12		21	27	56
5		37	26		2	11		37	0		42	9		36	11		21	6	55
6		38	26		1	51		37	59		41	48		37	10		20	44	54
7		39	26		1	31		38	58		41	28		38	9		20	22	53
8		40	25		1	12		39	58		41	7		39	8		20	1	52
9		41	25		0	52		40	57		40	47		40	7		19	39	51
10		42	25		0	33		41	56		40	26		41	6		19	17	50
11		43	25		0	13		42	56		40	6		42	5		18	56	49
12		44	24	56	59	54		43	55		39	45		43	4		18	34	48
13		45	24		59	34		44	54		39	24		44	3		18	13	47
14		46	24		59	15		45	54		39	4		45	2		17	51	46
15		47	23		58	55		46	53		38	43		46	1		17	29	45
16		48	23		58	35		47	52		38	22		47	0		17	8	44
17		49	23		58	15		48	52		38	2		47	59		16	46	43
18		50	22		57	56		49	51		37	41		48	58		16	24	42
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24		56	20		55	57		55	47		35	36		54	51		14	13	36
25		57	20		55	37		56	46		35	15		55	50		13	51	35
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27		59	19		54	57		58	44		34	33		57	48		13	7	33
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29		1	18		54	18	20	0	43		33	51		59	46		12	23	31
30		2	18		53	58		1	42		33	31	21	0	45		12	1	30
31		3	17		53	38		2	41		33	9		1	44		11	39	29
32		4	17		53	18		3	40		32	48		2	42		11	17	28
33		5	16		52	58		4	40		32	27		3	41		10	55	27
34		6	16		52	38		5	39		32	6		4	40		10	33	26
35		7	15		52	18		6	38		31	45		5	39		10	11	25
36		8	15		51	58		7	37		31	24		6	38		9	48	24
37		9	15		51	38		8	37		31	3		7	36		9	26	23
38		10	14		51	17		9	36		30	42		8	35		9	4	22
39		11	14		50	57		10	35		30	21		9	34		8	42	21
40		12	13		50	37		11	34		30	0		10	33		8	20	20
41		13	13		50	17		12	33		29	38		11	32		7	58	19
42		14	12		49	57		13	32		29	17		12	30		7	36	18
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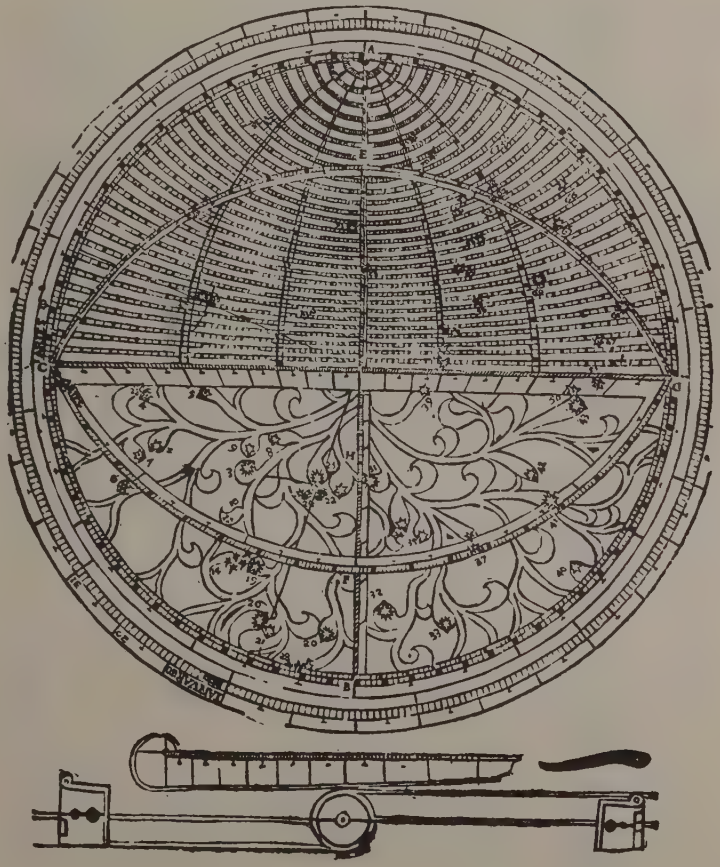
45		17	11		48	57		16	30		28	14		15	27		6	29	15
46		18	10		48	37		17	29		27	53		16	26		6	7	14
47		19	10		48	16		18	28		27	31		17	24		5	44	13
48		20	9		47	56		19	27		27	10		18	23		5	22	12
49		21	9		47	36		20	26		26	49		19	22		5	0	11
50		22	8		47	15		21	26		26	27		20	20		4	37	10
51		23	8		46	55		22	25		26	6		21	19		4	15	9
52		24	7		46	35		23	24		25	45		22	18		3	52	8
53		25	7		46	14		24	23		25	23		23	16		3	30	7
54		26	6		45	54		25	22		25	2		24	15		3	8	6
55		27	5		45	34		26	21		24	40		25	14		2	45	5
56		28	5		45	13		27	20		24	19		26	11		2	23	4
57		29	4		44	53		28	19		23	58		27	11		2	1	3
58		30	4		44	33		29	18		23	36		28	10		1	38	2
59		31	3		44	12		30	17		23	15		29	8		1	16	1

Plate 12. Tables from M. Bressius, Astronomicae libri quatuor, Paris, 1581.

THE MATHEMATICAL IEVVELL.

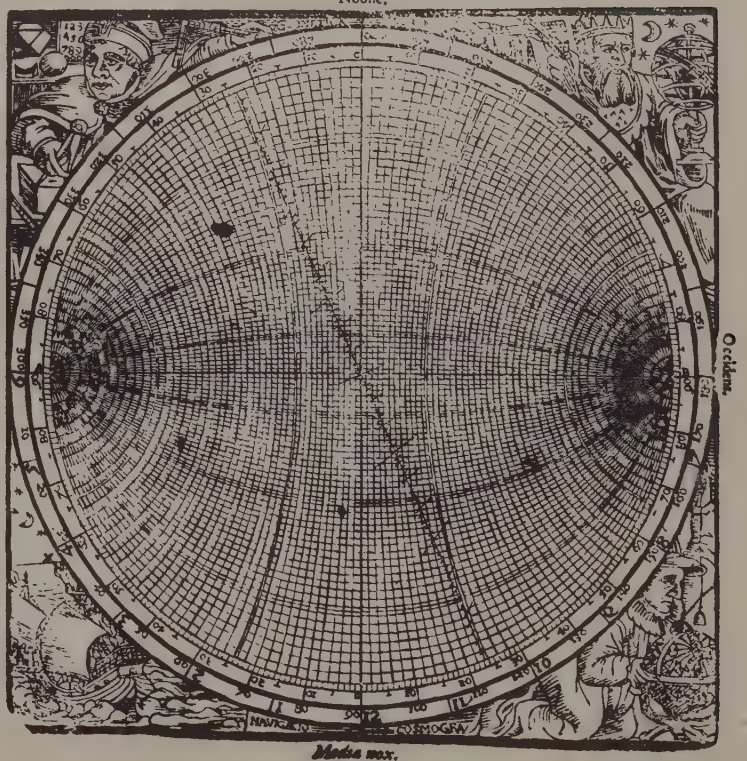


MARGARITA MATHEMATICA.

Aſtronomis nobiliſsimum, Geometris iucundiſsimum, Nauigantibus preſtantiſſimum, Coſmographis commodiſsimum, philoſophis, Medicis, & aliquid ſublimè affectantibus gratiſſimum, Tyronibuſq; facillimum.

This handle is here to be ſet on.

Meridies.
Noone.



Media nox.
North.

Per Ioannem Blagraum Readingenſem, conditum, editum, & Scu'ptum.
1584.

Plate 13. John Blagrave. The Mathematicall Jewell, 1584.

He computed his tables in a manner similar to those of Bressieu. His tables of sines, tangents and secants are computed for a radius equal to 10000000 for every minute from 0° to 90°.[92] Despite the fact that he had access to the tables of Rheticus, his own tables are not arranged for the reading of the complementary functions.[93]

The Law of Tangents occurs among his theorems for the plane triangle: "Ut semisses summae crurum ad differentiam summae semissis alteriusque cruris, sic tangens semissis anguli crurum exterioris ad tangentem anguli, quo minor interiorum semisse dicti reliqui minor est, aut major, major."[94]

The proof is given by von Braunmühl:[95]

Let $AE = \alpha$, $JE = \beta$, $AO = a$.

and $OJ = b$.

$AY = YJ = (\alpha + \beta)/2$.

$EY = YJ - JE$.

$JE = (\alpha + \beta)/2 - \beta = (\alpha - \beta)/2$

$$\left.\begin{aligned}\tan\,(\alpha + \beta)/2 &= AR\\ \tan\,(\alpha - \beta)/2 &= OR\end{aligned}\right\} \text{ for radius } UR = 1.$$

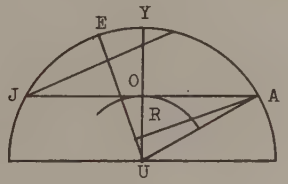


Fig. 110

but $AR = (AO + OJ)/2 = (a + b)/2$ and $OR = (AO - OJ)/2 = (a - b)/2$.

Hence upon dividing the tangent of the sum of the two arcs by the tangent of the difference the following result is obtained:

$$\frac{(a + b)/2}{(a - b)/2} = \frac{\tan\,(\alpha + \beta)/2}{\tan\,(\alpha - \beta)/2}.$$ [96]

This same formula was given in a more elegant form by Vieta in his treatise of 1593,[97] but not in his Canon mathematicus of 1579;[98] therefore, the Geometria rotundi is the first European text that contains a demonstration of this theorem.[99]

Finck had a great admiration for the work of Regiomontanus; he refers to it at the end of one of his exercises:[100]

Regiomontanus aliquot casus in secundo libro de triangulis collegit; quos illic videre potes. Cujus certe libri a studiosis avidi legi debent: et cum fructu legi possunt.	Regiomontanus collected several cases [of solution of triangles] in his second book of triangles which is accessible. His books ought to be read eagerly by students; they are able to be read with profit.

In his spherical trigonometry, Finck followed Bressieu; in one thing, however, he preceded Bressieu, that is in the statement of the Cosine Theorem, the form of which

92. Idem.
93. Idem.
94. Johann Tropfke, op. cit., p. 80.
95. Op. cit., p. 188.
96. Tropfke gives the following formula for Finck's expression of the Law of Tangents:
$$\frac{(a + b)/2}{\{(a - b)/2\} - b} = \frac{\tan\,(180^{\circ} - \alpha)}{\tan\,[\{(180^{\circ} - \alpha)/2\} - \beta]}.$$
97. See p. 85.
98. See p. 76.
99. In Proposition 21 of De triangulis libri quinque, Regiomontanus has a figure similar to that of Finck (see p. 30); however, in the proof of the accompanying theorem he uses only the sine function. This same figure is in the work of Copernicus (see p. 51).
100. A. von Braunmühl, op. cit., p. 188.

he took from Regiomontanus and applied it for the first time to the solution of a triangle when two sides and the included angle are given.[101]

Finck's Geometria rotundi was held in high regard by his contemporaries; it influenced, as the subsequent pages of this chapter will show, the works of Clavius, Lansberg, Magini, and Pitiscus.

6. JOHN BLAGRAVE

A work of John Blagrave,[102] The Mathematical Jewel printed in London in 1585, is not in the strict sense a trigonometric treatise nor does it advance any trigonometric notions. Its purpose, as the author explicitly states, is:[103]

> ...teaching most plainely to performe on the Iewell the whole art of spericall triangles, without any whit of that toyle by synes supplementes, tables, proportions, Arithmetical calculations, and such like: by which Ptolemeus, Regiomontan[us], Copernicus, and all other Authors before me, haue wrought them here performed by ocular inspection with surpassing facility, and maye bee called the key of knowledge of the sphere, in that it vnlocketh all spherical questions whatsoeuer: and is a singular introduction vnto the Sinicall working.

The mathematical jewel is defined by Blagrave as:[104]

> ...This instrument or any such like: some writers haue called Planispherium a plaine sphere, but the most part...Astrolabium or an astrolabe: the first springeth of the traduction or deriuation of the thing it selfe, founding as it were Sphera in plano a sphere or globe in plaine forme. The second commeth of the qualities or vse called, Astrolabium quia monstrat lapsus astrorum, bicause it sheweth the gliding or course of the starres. ...But I finding the vse of this instrument so infinite and vnspeakable, haue not without cause named it the Mathematicall Jewel.

In Chapter I, Blagrave gives his trigonometric definitions:[105]

> A quadrant is in briefe the quarter of a circle, encluded with two semidiameters, as you see here in the figure ABC. The arch thereof, vz. BC, I call the Limbe which is deuided as the order is, into 90. degr. The semidiameter erected, vz. BA is the side of the quadrant, the other semidiameter, vz. Ac is the groundline or base of the quadrant, from which the Limbe beginneth to be numbered, vz. from C toward B.
>
> An arch in Latine Arcus, is any porcion of the circumference or rynge of a circle, as here, BO, OP, PD, BC, etc.
>
> A Chorde in Latine Chorda, is as it were the diameter to the arche, or the string to the bow, as here the streight line BGP, is the chorde of arch BOP, and the line DC of the arch DC.

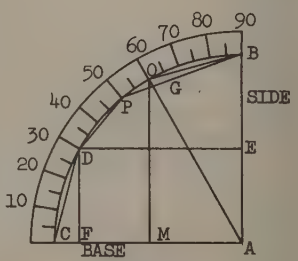


Fig. 111

> The Complement of an arch in Latine Complementum arcus, is that portion of a circle, which any arch lacketh of a quadrant or 90. degrees, as the complement of the arch BD is the arch DC, and the complement of DC, is BD. So that alwayes the arche and his complement make a quadrant.
>
> A Syne in Latine Sinus, is a streight line commensurable yet ther in length or power vnto the diameter of a circle, and to be measured by his parts.

101. Ibid., p. 189.
102. John Blagrave (d. 1611) was born at Reading, but the date of his birth is unknown. He received his early education in his native town, and afterwards entered St. John's College, Oxford, but did not take a degree (Dictionary of National Biography [London: Smith, Elder and Co., 1886], vol V, p. 157).
103. John Blagrave, The Mathematical Jewel (London: Walter Venge, 1585), p. 75.
104. Ibid., p. 20.
105. Ibid., pp. 75-76.

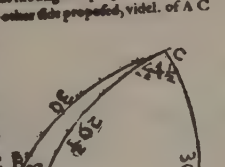
of the Globe, if otherwise, you of choice take. Also A I, A A, shall neerest hot the distance of the azimuth wherein the Sunne then was from the North, found by the 11. 130. deg. if out of that you take 90. there remaineth 40. deg. viz. B B, the distance of the same azimuth, from the East or West, and that towards the South. I am the briefer because I meane onely to give you a taste of the vse of those triangles, knowing this also to be vnto some so connexion, so shewshall vnderstand the 4. booke past euen by the bare inspection of this figure.

Chapter 13.

Ioh. Regiom. lib. 4. cap. 28. Copern. 11.

If of a spherical triangle two sides containing any one knowne angle shalbe giuen, how to finde the other side and angles.

LEt there be propoſed the ſpheriall triangle A B C, whoſe angle A, let it be knowne 58½ degrees togither with the two ſides, containing it viz. A C 36. degrees, and A B 25. Reckon the quantitie of the one side giuen, as of A B, 25. deg. on the *Limbe* of your *Iewel*, thereto lay the *Rewle* (zenith fixed: then on the meridian diſtant from the *Limbe*, as much as the angle A propoſed commeth to videl. on the 58½ meridian, reckon from the pole the quantitie of the other ſide propoſed, videl. of A C 36. degrees, and looke what azimuth croſſeth the ſame 58½ meridian in the ſame 36. deg. the ſame followed to the *Limbe*, ſheweth the degrees of oneof the angles ſought for, videlicet of the angle B ſubtended by the ſide 36. which you ſhall finde 82½ degrees counted from the *Limbe*: then are the degrees of the ſame 82½ azimuth betweene the ſaide croſsing and the zenith, the quantitie of the third ſide ſought for, which you ſhall finde to be 30. deg. Thus doe you ſee according to your accuſtomed manner your whole triangle in his forme, his three ſides, & two angles knowne, wherefore by the 11. Chapter the third angle is eaſily haden the laſt part of which Chapter, this Chapter was performed ere I was aware, ſuch is the dexterite of my *Iewel*, neither wouldst I haue repeated it here againe, had it not beene to imitate the courſe of Regiomontanus and Copernicus.

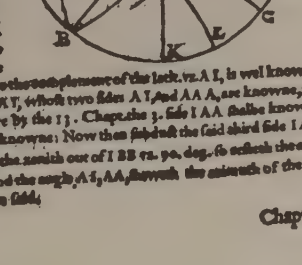


In briefe, lay the one ſide giuen betweene the pole and the zenith, the other reckon on the meridian as much diſtant from the *Limbe*, as the angle giuen commeth to, there ſtad moare you the azimuth cutting vp your triangle.

Chapter 14.

How by helpe of the 13. Chapt. to know the height of the Sunne at any houre, and minute propoſed on the whole yeare, and alſo at what azimuth he is.

IN my thirde booke 21. Chapter is ſhewed the ſame by ordinary vſe of the *Iewel*, and that moſt exact and ruddily without any trouble, yet ſhall it not be amiſſe to know to do the ſame by the Art of ſpheriall triangles, which performeth it as reddily as Gem. Friſius did on his Catholicon, though nothing ſo eaſie as my *Iewel* doth it in the ſame 3 Chapt.



For example the 28. of Iune, the ſun then being in the 1. degree of ♋ his declination there 23½ deg. I would know of what height, and in what azimuth the ſun ſhall then be. The angle made betweene the 10. of the clocke houreline, and the meridian being in this figure, the angle A A, A I, muſt needs be 30. deg. for that every houre is 15. degrees, then taking the ſunnes declination for thatday viz. D N, or rather C C A A 23½ deg. out of 90. deg. videl. C C A, there reſteth the arch of the 10. of clocke houreline betweene the tropicke or ſunnes declination, and the pole viz. A A, A being 66½ degr. Then the portion of the meridian betweene the pole and the zenith, being always the complement of the lack. viz. A I, is wel knowne 38½ deg. Here then haue you a ſpherical triangle A A, A I, whoſe two ſides A I, And A A, are knowne, including the knowne angle A A, A I viz. 30. deg. wherefore by the 13. Chapt. the 3. ſide I A A ſhalbe knowne 36½ deg. and the angles A I, A A & A A, A I, ſhal alſo be knowne: Now then ſubſtract the ſaid third ſide I A A viz. 36½ deg. being then the diſtance of the ſunne from the zenith out of I B B viz. 90. deg. ſo reſteth the arch A A, B 53½ deg. the altitude of the ſunne ſought for, and the angle A I, A A, ſheweth the azimuth of the ſun in the very ſame manner, as in the 12. Chapt. hath beene ſaid.

Chapter

Chapter 15.

Ioannes Regiomontanus, Lib. 2. Cap. 51. and lib. 4. cap. 29. *propoſeth and ſayth that the knowing of two ſides of a triangle not right angled, with one angle ſubtended by one of thoſe ſides, cannot ſerue to get the other ſide and angles.*

HEere is a wonderfull caſe, that by their whole art of ſpearicall triangles, with their ſignes, coardes, arches complements, proportions, calculations, tables, and ſuch like, they cannot doe that which my poore *Iewel* will doe immediatly without any difficultie. Nay all the ſporte is, Regiomontanus ſpendeth muche ſpeache in the ſaide 51. and 29. chapter, to prooue that it is vnpoſſible to bee done, without letting downe a perpendiculer, reſpecting or ſubtending the angle propoſed, which is the therrieth chapeer following, hee teacheth to doe. But we will doe it, add that without any perpendiculers, a little readier then he doth it.

The figure of the 11. Chap. ſerueth for this place.

Admit that of the triangle A B C, propoſed in the thirteenth chapter, the two ſides A B, and A C, were knowen as there they were, togeather with one of the angles B or C, ſubtended by one of the ſides giuen: for example, the angle B 82½ degrees, ſubtended by A C, & that the angles A and C were vnknowen, with the ſide B C: Is it not poſſible thinke you by theſe three giuen, to reache to reſt, I hope it be, els I looſe my labour.

Reck ō on the *limbe* of the *Iewell* frō the pole ♑ ſide giuen, not ſtbeandyng the angle giuen, !vz. ♑ ſide A B, being 25. degrees, thereto lay the neereth point of the *reere* fixed, reckon alſo on the *Fiwere* from the *Limbe* inwards, the angle B giue, vz. 82½. de. there haue you the 82½. azimuth, then reckon the other ſide giuen, vz. A C. 36. degrees among the parallels, counting from the pole, mark where this 36. parallel, and the ſaid 82½. azimuth doe croſſe. For the portion of the meridian cutting there, counted too the pole, is the thirde ſide, which you ſhall finde 30. degrees, of the 58½. meridian counted from the *limbe*. The ſame 58½. meridian followed alſo to the equinoct. ſheweth 58½. deg. one of the angles ſought for, vz. A. The third angle C cannot bevnknowen by the 11. or 13. Cha. Thus you ſee this vnpoſſible matter now poſſible, then iudge whether my *Iewel* bee a *Iewell* or no.!

Yet thus much will I foretell you, leaſt happily you ſhoulde be amazed at any time. In ſome triangles you ſhall finde that the azimuth repreſenting the angle giuen, will croſſe the parallel that worketh this feate in two places, and then ſhall you not tell which place of croſsing to take, yet commonly the croſsing fartheſt within the *Limbe* is hit: in all the triangles that I haue had experience of, but you may trie by ſubſtung of the ſides, as you are wont to doe in finding the thirde angle which is it.

Chapter. 16.

The azimuth of the Sunne giuen or taken at any time by Inſtrument, with his declination, how by helpe of the fifteenth Chapter, to knowe what height the Sunne is of, and what it is a clocke.

IT is taught in the 12 chapter, ſpherically to get the houre & azimuth of the Sunne by his altitude, contrarwiſe, heere by the azimuth ſhall bee hadde the houre and the altitude.

The figure of the 5 chap. ſerueth for this place.

For example, the 12. of Iune, admit that before noone I had take by ſome Topographicall Inſtrumēt, or otherwiſe the azimuth or horrizontall diſtance of the Sunne 40. degrees from the Eaſt towardes the South, which in this figure is O B B, thereto I adde 90. degrees, videlicet, B O: ſo haue I the arche *E B*? 130. deg. the declination of the Sunne then is 23½. degrees, by the 3. booke and 4. chapter. Now it is moſt manifeſt, and apparantly to bee ſeene on the *Iewel*, that any parallell of the ſpheare doth cut off the meridians, or houre lines, alſo like equall diſtance from the pole, as heere you ſhall ſee the arches of the houre lines A X. A Z. A AA, A N, are all complements of the Sunnes declination: each of them, and namely A AA, 66½. beeing 66½. degrees, the arche betweene the pole and the Zeaith, viz. A I, is certaine 38½. degrees. Here haue you nowe a ſpherical triangle A I, A A, whoſe two ſides A I, and A AA, are knowen with the angle A I, A A, which angle is ſubtended by A, AA, one of the knowen ſides, euen the like too that whiche the fifteene chapter propoſeth. The third I, AA, and the angles AA, A I, and A, AA I, are vnknowen. VVherfore if the worlde were ſo harde, and the matter ſo vnpoſſible, as Regiomentanus propoſeth it, we ſhoulde be neuer the neere of our purpoſe: but working by the 15. chapter, notwithſtanding his proofes too the contrary, you ſhall finde the angle *I, AA*, 39. degrees, the Sunnes houre diſtaunce from the South, which muſt needes be two houres euen ten of the clocke: alſo you ſhall finde the thirde ſide, *I, AA*. 36½. degrees, whiche taken out of 90. *vz. I, BB*, leaueth the arch *AA BB*, 53½. degrees, the height of the Sun at that inſtant. As for the thirde angle, you ſhall finde him 32. degrees, but wee haue no vſe of him here.

Plate 14. Page from Blagrave, Mathematicall Jewell.

NICOLAI RAYMARI VRSI DITHMARSI.

FVNDAMENTVM ASTRONOMICVM:

ID EST.

NOVA DOCTRINA SINVVM ET TRIANGVLORVM.

EAQVE ABSOLVTISSIMA ET PERFECTISSIMA, EIVSQVE VSVS IN ASTRONOMICA Calculatione & Obſeruatione.

Cui adiunctæ ſunt:

I. Hypotheſes nouæ ac veræ motuum corporum Mundanorum.
II. Extructio Canonis ſinuum, vulgaris quidem, ſed ſolitâ viâ facilior.
III. Sectio anguli datâ ratione, ſeu in quotlibet partes.
IV. Quadratio Circuli Demonſtrabili ratione, eiusque Demonſtratio.
V. Solutio Triangulorum vſitatiorum nouâ ac facillimâ ratione.
VI. Solutio plerorumque Triangulorum per ſolam proſthaphæreſin.
VII. Eiuſdem proſthaphæreſeos Apodixis, Cauſa, ac Demonſtratio.

Omnibus ſeculis problemata ſanè deſideratiſſima, aliaq; priùs nec audita nec anteà viſa paradoxa quàm plurima.

Αγεωμέτρητος μηδεὶς εἰσίτω.

ARGENTORATI.
Excudebat Bernhardus Iobin.
1588.

Plate 15. Nicolas Raimar Ursus. Fundamentum Astronomicum. Strassburg, 1588.

The Syne totall in Latine Sinus totus, is the side or base of the quadrant, as AB, or AC, divided into an 100. equal parts at the least, but rather into a 1000. 10000. or 100000. parts for the more the better.

A right Syne in Latine Sinus rectus, is anye perpendicular falling from the arche or Limbe of the quadrant on his base, as here you see DF is the right Syne of the arch DC and OM of OC, and is denominated of those parts of the Syne total whiche he contayneth or more briefly a right Syne is halfe the cord of an arch doubled, as the right Syne of BO is the line BG.

A backe Syne in Latine Sinus versus, is anye perpendicular falling from the arche or Limbe of the quadrant on his side, as you see DE, which is called the backe sine of the arch DC. By this you see and learne if you marke it, that the backe Syne of any arch, is the right Syne of the same arches complement, for DE is the backe Syne of the arch DC, but is the right Syne of BD. Wherefore in briefe, a backe Syne is halfe the chorde of the arches complement doubled.

The Supplement of a Syne is that which any Syne lacketh of the quadrant side or Syne totall, as BE is the supplement of the right Syne FD, or AE, which is equal vnto it: likewise FC is the supplement of the backe Syne ED, for ED and FC make the Syne totall.

Note that some Authors make but three distinctions of Synes as I haue here done, vz. totus, rectus & versus. Others some as namely Apian and Munster[106] make foure, vz. Sinus totus, then Sinus rectus primus, which here I call the right Syne, and Sinus rectus secundus, which I call the backe Syne, and then Sinus versus to be what I call the supplement of the backe Syne, videl. FC, being as the arrow to the bow; but truly I like not thereof, and therefore I haue thought good to call both FC and EB Supplements as before, for if so be that FC be Sinus versus, then may ED as wel be Sinus versus secundus.

A portion of Chapter II is devoted to the definitions of spherical geometry which Blagrave says he obtained from Book IV of the treatise of Regiomontanus.

Blagrave solves his spherical triangles by means of his "jewel" and does not use the trigonometric functions which he has defined. In marginal notes he states from what portions of the treatises of Regiomontanus and Copernicus he has obtained his theorems.

In so far as trigonometry is concerned, Blagrave's work is inferior to that of his contemporaries; however, his purpose justifies his treatise.

7. CHRISTOPHER CLAVIUS

Among some of his contemporaries Christopher Clavius was known as the "Euclid of the sixteenth century."[107] Others of his time, contesting his right to this title, felt more inclined to agree with Vieta's statement that he was a good professor who was able to explain with clarity the inventions of other people.[108] True, the works of Clavius do show dependence on other works, and while his contributions to the development of trigonometry are of a much lesser merit than those of Vieta, yet their consideration is essential to the history of the development of trigonometry.

The trigonometric contributions of Clavius can be found in two of his works, namely, Theodosii Tripolitae sphaericorum libri III. Item sinus, liniae, tangentes et secantes, triangula rectilinea atque sphaerica...which was published at Rome in 1586[109] and Geometrica practica which was published in Rome in 1604.[110]

In the preface to the first named work, Clavius has a reference to two editions of Theodosius that were published in 1558, the one being John Pena's Latin translation of the Greek and the other Maurolyco's Latin translation of the Arabic.

106. Blagrave refers to Sebastian Münster (1489-1552) who was professor of theology, Hebrew, and mathematics at the University of Basle (Poggondorff, op. cit., vol. II, p. 232).
107. Jean Mascart, "Clavius et l'Astrolabe," extract from Bulletin astronomique, Paris, 1905, p. 4.
108. Idem.
109. This work in the Opera omnia published at Mainz in 1611 was used.
110. The edition published at Mainz in 1606 was used.

The treatise of 1586 contains tables of sines, tangents and secants calculated for a radius equal to 10000000. The tables in the Geometrica practica, according to Braunmühl,[111] are similar to those of Finck except that they consider the complementary functions. In his definitions Clavius uses the terms "sinus rectus," "sinus versus," and "sinus complementi," although he states that for the first and last function the expressions "sinus rectus primus" and "sinus rectus secundus" are also used. He follows Finck in his use of the terms "tangent" and "secant," but refers also to the expressions "adscripta" and "prosinus" for the former and "hypotenusa" and "transsinuosa" for the latter.

His theorems imply such relations as this of Theorem 2, Proposition 4, of his treatise on the sine function preceding the table of sines in the commentary on Theodosius:

> ...sinus totus ad duplum sinus dati eandem proportionem habet, quam datus sinus ad sinum versum dupli arcus.

$$\frac{1}{2 \sin a} = \frac{\sin a}{\text{vers } \sin 2a}$$

Let $a = b/2$ and the formula becomes the familiar one:

$$2 \sin^2 b/2 = 1 - \cos b.$$

And this one of Theorem 4, Proposition 6: "The chord of the difference between two arcs of a semicircle of which one is less than 120° by a certain amount and the other is greater than 120° by the same amount is equal to the chord of an arc which differs from either arc by 120°.

Clavius constructs Fig. 112 in the following manner:

In the semicircle whose diameter is BC, he constructs the chords BD and BE whose arcs differ from the arc BA, which is an arc of 120°, by an amount x. On BE he lays off the length BF equal to BD; he proves triangles ADB and AFB congruent, whence DA equals AF. But DA equals AE by construction; in the triangle FAE angle AEB is an angle of 60°, hence the other base angle equals 60°, therefore the triangle is equilateral and FE equals AE. Therefore the difference between the chords of the two arcs BD and BE is equal to the chord of the arc x. This theorem implicitly contains the formula:

$$\sin (60 + x/2) - \sin (60 - x/2) = \sin x/2.$$

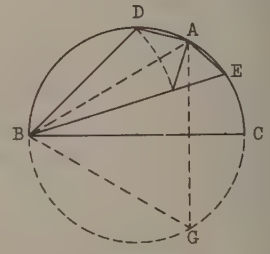


Fig. 112

The theorems that conclude the section of the treatise on the sine are similar to the propositions of the Tractatus of George Peurbach.[112]

Theorem 10, Proposition 18, contains some of the fundamental identities pertaining to the tangent and secant functions, namely:

> The ratio of the cosine of any arc to the sine is as the total sine to the tangent of this same arc, and the ratio of the sine to the cosine is as the total sine to the cotangent. The total sine is the mean proportional between the cosine and secant of the same arc; also between the sine and cosecant. The total sine is also the mean proportional between the tangent and cotangent of the same arc.

Clavius' treatise on plane triangles is short and precise; each theorem is accompanied by a numerical illustration. In the preface to this section of his work, Clavius refers to his predecessors, mentioning the names of Menelaus, Geber, Regiomontanus, Copernicus, and Maurolyco. The text proper indicates his familiarity with the works of Vieta and Finck.

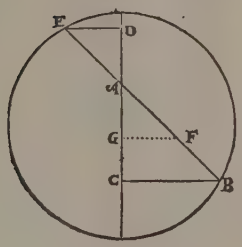
111. Op. cit., p. 189.
112. See pp. 31-32.

beatur portio minor ipsius secandi anguli, qui & in hunc modum dat á ratione sectus erit. Vel aliter sic: notis tribus lateribus Trianguli Rectanguli AMN, erit vt AN ad Radium seu sinum anguli Recti AMN, sic MN ad IO, sinum nempè anguli quærendi MAN seu HAI. Idq; per 9. id quod idem est ac si dixissem: vt Radij AI segmentum AN ad ipsum Radium totum AI, ita NM ad IO, per 6. seu 2. sexti.

Atq; insuper præmittendum est ad sequentem solutionem & hoc lemma seu assumptum, quod est consectarium dependens è 6.

Ductæ parallelæ è terminis alterius Rectæ in reliquam sese obliquè intersecantium, sunt comprehensis inter se segmentis vtriusq; Rectæ, oppositè seu alternè proportionales.

Diagramma lemmatis.



Declaratio: Dico in adiuncto Diagrammate esse vt AB ad BC, sic AF seu AE ad FG seu ED. per 6.

Sed iam omnia, quorum machinationibus ad huius perplexi atq; intricati nodi Ergastulum opus erit, in promptu ac præcinctu esse videntur, ideoq; iam lato celeusmate ipsam soluendi nodi molem aggrediamur.

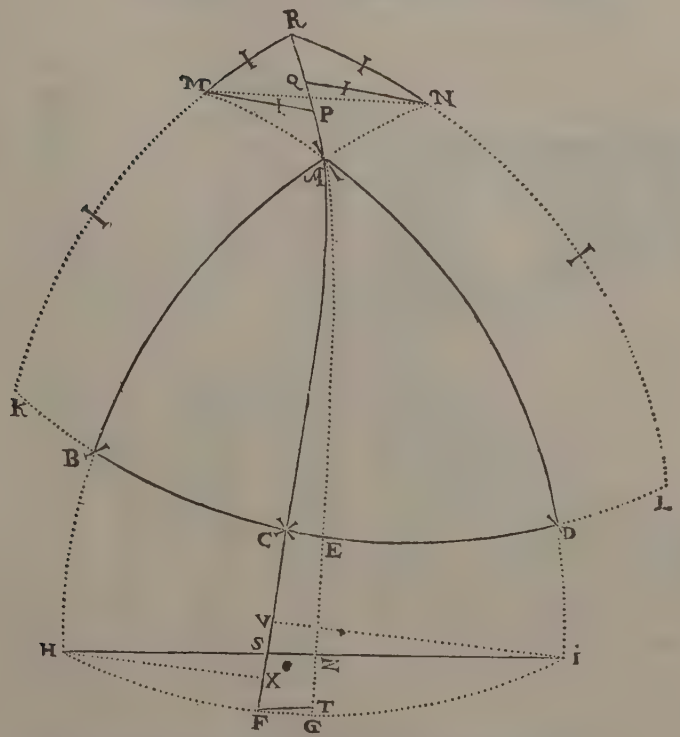
Delie iò pæan, & iò eüie pæan.

VI. Datis omnibus Trianguli obliquanguli sphærici angulis, inuestigare latera.

Dia-

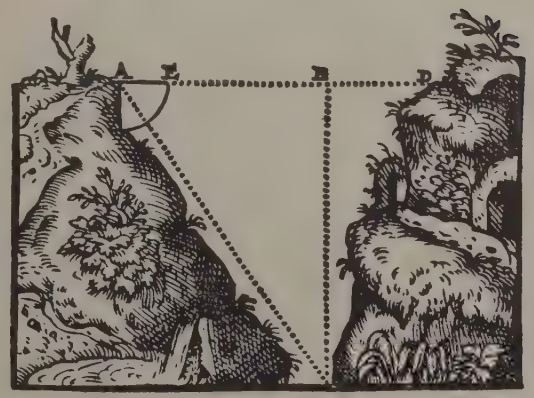
Diagramma vltimi Obliquanguli. Christophoro Clauio Bambergensi donatum.

SALVE VENERANDE SACERDOS.



Soluendum Triangulum ABD. Eiusq; dati Anguli ad ABD. Latera vtrobiq; continuata vsq; ad completionem quadrantium, idq; tùm dextrorsùm in L, tùm sinistrorsùm

Plate 16. Illustration from Nicolas Raimar Ursus, 1588.



superficies A C declinat ab horizontis plano, ex eo namq; , & latere AC superius cognito colligere poteris promptissimè profunditatem BC per primum casum 13. plan. triang.

Exemplum.

Esto angulus B A C grad. 48,& sit explorata distantia A C pedum 50. Ex his igitur innotescet profunditas B C pedum 37,cum septima parte.

PROPOSITIO VI.

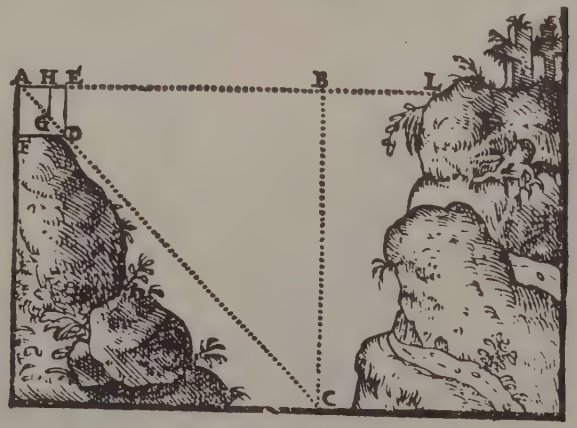
Eandem profunditatem eo, quo dictum est, modo per Quadratum colligere.

Cape distantiam terminorum A, & C per tertiam primi huius,& firma eo Quadrato ad libellam in termino A respice infimum C signando partes, quas dioptra intersecat, quæ erunt vel in latere recto, vel in verso, vel in linea Quadrati media.

PRIMVS CASVS, *quando cadit regula in linea diagonali ipsius Quadrati.*

Occurrat primo regulæ intersecatio in diagonio A D ipsius Quadrati, tunc quæ proportio est partium ipsius regulæ A D ad totum Quadrati latus,

tus'eadem erit distantiæ A C notæ ad incognitam profunditatem B C, propter æquiangula triangula E D A, B C A, vt patet. Partes vero ipsius A D sunt semper 141, cum 21. quinquagesimis, seu cum tribus septimis.



Exemplum.

Demus in exemplo adinuentam fuisse distantiam A C pedum 40. Hos igitur numeros proportionales ordino sic pro elicienda profunditate B C.

1	2	3	4
Vt partes AD	ad partes E D	ita distantia	ad profunditatem
141 21 / 50	100	ac pedum 40	B C pedum 28 9/60

ALITER.

Prôptius etiam colligi potest ipsa profunditas absque notitia partium A D. Nam cum sit eadé, proportio B C ad B A, quæ est E D ad E A, sintq; duo Quadrati latera E D, E A inter se æqualia, erunt etiam æquales rectæ B C, BA, quorum quadrata æquabuntur quadrato rectæ A C, per 47 primi elem. Conficiatur ergo quadratum ipsius A C, nam eius dimidium 56 dabit

Plate 17. G. A. Magini. De planis triangulis, Liber unicus, 1592.

Clavius' first theorem is the statement of the Sine Law for a plane triangle. He proves the theorem first for a right triangle. For a scalene triangle, obtuse or acute, he constructs figures similar to Fig. 113(a) and Fig. 113(b) by assuming ABC to be the given triangle and making CE equal to AB and drawing AD perpendicular to BC. This is essentially the same as the method employed by Al-Biruni and Regiomontanus.[113]

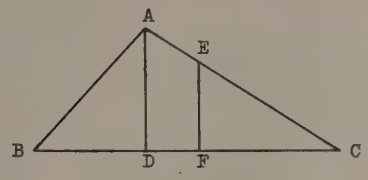


Fig. 113(a)

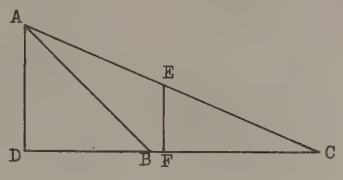


Fig. 113(b)

The sine law is applied to this numerical illustration: "Given one side with one acute angle of a right triangle or with the ratio of two of the angles, the remaining two sides will be known."

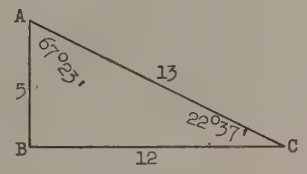
AC.	AC.					
100000	13	AB.	will be	AB.	
		38456?		5	
		BC.		BC.	
		92310?		12	

Fig. 114

Total sine equals 100000, sine of angle opposite AB is 38456, sine of angle opposite BC is 92310; the application of the sine law gives the value 5 to the side AB and 12 to BC.

The other cases for the solution of a plane right triangle are considered in the same manner; the dimensions of Fig. 114 are used throughout.

Clavius' work contains another theorem that has occurred in previous texts: "If the diameter of a circle divides a chord into two segments, the segments will have the same ratio as the sines of the respective arcs."

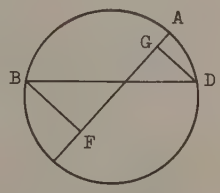


Fig. 115

The following theorem is taken from Regiomontanus: "If from an angle of a triangle with known sides, a perpendicular is drawn to the opposite side, the portions of the side divided by the perpendicular, and the angles will be known." The figures illustrate the two methods of proof that are used.

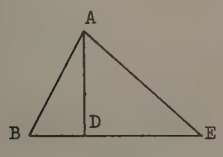


Fig. 116(a)

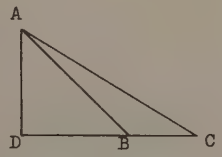


Fig. 116(b)

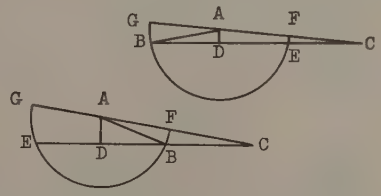


Fig. 117

113. See p. 9. Nicolas Ursius also has this proof in his Fundamentum Astronomicum published in 1588.

Clavius reduces all the cases of the solution of an oblique triangle to the solution of right triangles, that is, all except the case for two sides and the included angle, for which he uses the tangent theorem in this form:

$$\frac{\frac{a+b}{2}}{\tan\frac{(A+B)}{2}} = \frac{\frac{a+b}{2}-b}{\tan\frac{(A+B)}{2}-B}$$

thereby imitating Finck who was the first to use this theorem for the same case of the solution of an oblique triangle.

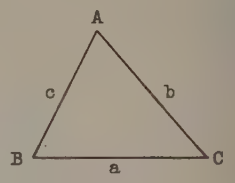


Fig. 118

In many of the numerical illustrations there is a reference to the "Golden Rule" as the method of solution; the "Golden Rule" is Clavius' expression for the rule of proportion.

The spherical trigonometry contained in his treatise of 1586 offers nothing new.[114] Like Regiomontanus, Clavius preludes the trigonometric portion of his work with a brief exposition of spherical geometry. Most of his marginal notes refer to the work of Theodosius. The errors he detected in the trigonometry of Copernicus have already been mentioned.[115]

The plane trigonometry contained in his later work of 1604 under the heading "Problemata varia triangulorum rectilineorum" of Chapter III, Book I, shows no advance over his former work, except that a simplified arrangement is helpful to the reader. In this treatise the tangent theorem is given in this form:

$$\frac{\frac{a+b}{2}}{\tan\frac{(180-C)}{2}} = \frac{\frac{a+b}{2}-b}{\tan\left[\frac{(180-C)}{2}-B\right]}$$

The second book of the Geometrica practica contains twenty-four problems whose solutions are obtained by means of a quadrant[116] and trigonometric formulas. Problem XX with its accompanying figure is representative. The problem is to estimate the height of a tower which is built on a mountain. The size of the angles are found by means of the quadrant; the formula used to calculate the required height is:

$$\frac{\sin BAD \text{ or } \cos ADC}{DB} = \frac{\sin ADB \text{ or } \sin (ADC - CDB)}{AB}$$

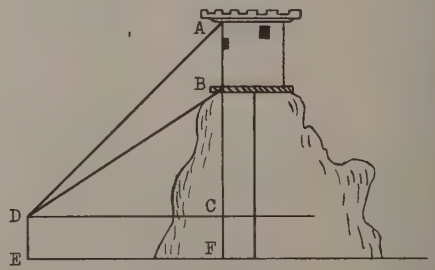


Fig. 119

Clavius' trigonometric treatises give evidence of his detailed study of the work of his predecessors. His systematic exposition of the theorems and their application was imitated by his successors.

8. PHILIPP VAN LANSBERG

Philipp van Lansberg[117] seems to have followed Finck more closely than he followed any other contemporary writer on trigonometry. In his Triangulorum geometriae

114. The contributions of Clavius and others to the study of stereographic projection will not be considered in this thesis.

115. See pp. 49, 51, 52.

116. The quadrant, an instrument for measuring altitudes, consists of a graduated arc of 90° with an index or vernier, and usually has a plumb line for finding the vertical or horizontal direction.

117. Lansberg (1561-1632) was born at Ghent and supervised the reform of studies at Antwerp; later he went to Goes in Zeland. In 1614 he devoted his time to mathematics and astronomy at Middelburgh. He enjoyed the reputation of being an outstanding astronomer (A. von Braunmühl, op. cit., p. 192).

IO.

ANTONII MAGINI PATAVINI

MATHEMATICARVM IN ALMO

Bononiensi Gymnasio professoris.

DE PLANIS TRIANGVLIS Liber Vnicus.

Eiusdem

DE DIMETIENDI RATIONE

per Quadrantem, & Geometricum Quadratum,

LIBRI QVINQVE.

Opus valdè vtile Geometris, Astronomis, Geographis, Mechanicis, Architectis, Militibus, Agrorum mensoribus, & denique omnibus Mathematicarum professoribus.

CVM PRIVILEGIO.

VENETIIS, APVD ROBERTVM MEIETTVM.
M. D. XCII.

PROPOSITIO VII.

Si ab angulo trianguli scaleni notorum laterum in maximum latus oppositum perpendicularis demittatur, quantum sit vtrumq; basis segmentum à perpendiculari factum inquirere.

Assumatur triangulum præcedentis propositionis, in quo quidem cum dentur duo latera minora A B, A C, nota erũt eorum quadrata, nec non etiam eorundem differentia, hoc est rectangulum sub basi B C, & differentia segmentorũ, quæ est E C, cumq; detur B C, cognoscetur quoque E C diuidendo ipsum rectangulum per notum latus B C. Dempta autem E C à toto latere B C dato, relinquitur B E nota, ipsiusq; dimidia B D, nempe minus segmentum; addita vero D E, hoc est altera dimidia ipsius B E, ad E C cognitam, constabit alterum segmentum maius D C.

Exemplum.

Vt in assumpto triangulo A B C, cuius latus A C est 15. latus A B 10, & basis B C 18. subtraho quadratum lateris A B, nempe 100. à quadrato lateris A C 225. & relinquitur 125. rectangulum scilicet contentum sub basi B C, & E C differentia segmentorum, quod quidem per ipsam basim diuisum dat in quotiente. 7. ferè, & tanta est E C segmentorum differentia, hæc autem sublata à tota basi B C relinquitur 11. pro linea B E, cuius dimidiũ 5. & semis est minus segmentum B D, & proinde maius quoq; segmentum D C erit 12. & semis.

Aliter.

Sit sicut supra perpedicularis A D cadens ab angulo A in maximum latus B C, faciens segmenta B D, D C. Cum igitur ambo anguli ad D recti sint, erit vterq; angulus B, & C acutus iuxta 17. primi elem. & per 13. secundi erit quadratum lateris A C minus, quàm quadrata duorum laterũ A B, B C circa angulum acutum B rectangulo bis comprehensio sub B C, & B D; cum autem dentur singula latera, dantur etiam eorum quadrata, à coniunctis ergo quadratis A B, B C, detractum quadratum A C relinquitur rectangulum bis contentum sub B C, B D. cuius dimidium diuisum per latus B C notum, dabit ipsum minus segmentum B D. Similiter etiam si quadratum A B auferatur à summa duorum quadratorum A C, B C, relinquetur duplex rectangulum sub B C, & D C, cuius quidem dimidium per latus B C diuisum, producet ipsum D C maius segmentum

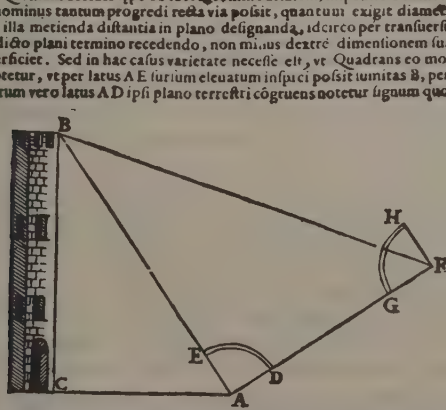
Exemplum.

Quoniam latus A B est 10. erit eius quadratum 100. & latus B C 18. dabit quadratum 324. quare horum duorum quadratorum summa erit 424.

B D A, D B A æquales. Cum vero angulus internus B D A sit ex operatione ipsa dimidium anguli externi B A C, reliquus etiam internus A B D erit eiusdem anguli B A C externi dimidium, ideoq; æquales erunt duo anguli B D A, A B D, & proinde duo etiam latera illos respicientia, quæ sunt A D, A B ex sexta primi elem. inter se æquabuntur. Quod demonstrandum fuit.

Secvndvs Modvs.

Quoniam accidit sæpe ob loci incommoditatem, impediri Geometrã, quominus tantum progredi recta via possit, quantum exigit diametralis illa metienda distantia in plano designanda, idcirco per transuersum à dicto plani termino recedendo, non minus dextrè dimentionem suam perficiet. Sed in hac casus varietate necesse est, vt Quadrans eo modo apetur, vt per latus A E furium eleuatum inspici possit umitas B, per alterum verò latus A D ipsi plano terrestri cõgruens notetur signum quod-



piam G in eius directo iacens. Quo facto denuo Quadrans disponatur, vt latus F G sit in rectum signorum G, & A, in quo quidem recto itinere exploretur situs, quo rursum cacumen B per regulæ foramina intersecantis 45. gradum appareat. Hoc artificio colligere poteris distãtiam signo tam A B, omensio scilicet interuallo duorum locorum F, & A illi æquali.

Demonstratio.

Cum in triangulo F B A angulus B A F factus sit per Quadrantem rectus, & angulus B F G sit gr. 45. erit per 32. primi elem. totidem gradus alter

alter acutus F B A. Quare per 6. primi elem. duo latera eis opposita F A, A B erunt æqualia. Quod demonstrandum erat.

PROPOSITIO XI.

Eandem distantiam diametralem signi in plano positi à signo quopiã in altum sito ædificij perpendiculariter ad illud planum erecti, ita tamen, vt & ad signum plani, & ad basim ædificij accedi possit, per Quadratum Geometricum indagare.

Poteris vero per quadratum Geometricum distantiam diametralem A B in hunc modum venari. Posito Quadrato ad perpendiculum observabis signum B per dioptram, & notabis punctum ab ea sectum, quod quidem erit vel in latere D E vmbræ rectæ, vel in linea A E vmbræ mediæ, vel deniq; in latere E F vmbræ versæ.

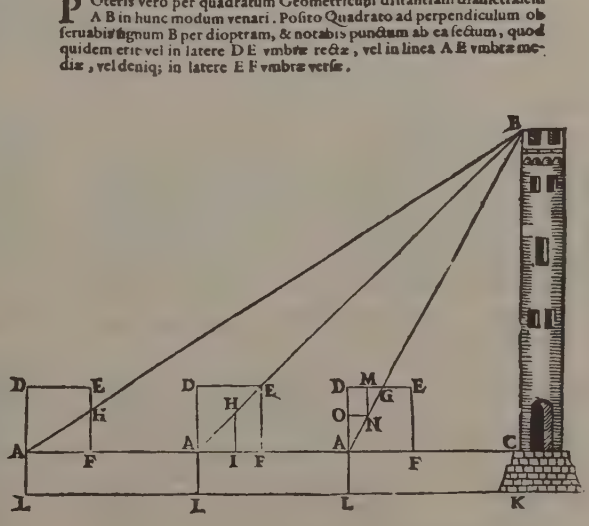


Plate 18. G. A. Magini. De planis triangulis, Liber unicus, 1592.

	42		43		44		
	Prima	Secunda	Prima	Secunda	Prima	Secunda	
0	90040,40	111061,24	93251,51	107236,86	96568,88	103553,02	60
1	90093,08	110996,29	93305,91	107174,34	96625,11	103492,76	59
2	90145,79	110931,38	93360,34	107111,86	96681,37	103432,53	58
3	90198,53	110866,52	93414,80	107049,42	96737,66	103372,34	57
4	90251,30	110801,70	93469,29	106987,02	96793,98	103313 19	56
5	90304,10	110736,92	93523,81	106924,66	96850,34	103252,07	55
6	90356,93	110672,18	93578,35	106862,33	96906,74	103191,99	54
7	90409,78	110607,48	93632,92	106800,04	96963,15	103131,94	53
8	90462,66	110542,83	93687,52	106737,79	97019,60	103071,93	52
9	90515,57	110478,22	93742,15	106675,58	97076,09	103011,96	51
10	90568,50	110413,65	93796,82	106613,41	97132,61	102952,02	50
11	90621,46	110349,12	93851,52	106551,28	97189,16	102892,12	49
12	90674,45	110284,63	93906,25	106489,19	97245,74	102832,25	48
13	90727,47	110220 28	93961,01	106427,13	97302,35	102772,42	47
14	90780,52	110155,77	94015,80	106365,11	97359,00	102712,62	46
15	90833,60	110091,40	94070,62	106303,13	97415,68	102652,86	45
16	90886,70	110027,08	94125,47	106241,19	97472,39	102593,14	44
17	90939,83	109962,80	94180,34	106179,29	97529,13	102533,45	43
18	90992 99	109898,56	94235,24	106117,42	97585,91	102473,80	42
19	91046,18	109834,36	94290,17	106055,59	97642,72	102414,18	41
20	91099,40	109770,20	94345,13	105992,80	97699,56	102354,60	40
21	91152,65	109706,08	94400,12	105932,05	97756,43	102295,06	39
22	91205,93	109642,00	94455,14	105870,34	97813,34	102235,55	38
23	91259,23	109577,95	94510,19	105808,67	97870,28	102176,07	37
24	91312,56	109513,96	94565,28	105747,03	97927,25	102116,63	36
25	91365,92	109450,00	94620,40	105685,43	97984,25	102057,22	35
26	91419,30	109386,08	94675,55	105623,87	98041,28	101997,85	34
27	91472 71	109322,21	94730,73	105562,35	98098,35	101938,52	33
28	91526,15	109258,38	94785 94	105500,87	98155,45	101879,22	32
29	91579,62	109194,59	94841,18	105439,42	98212,58	101819,96	31
30	91633,12	109130,84	94896,45	105378,01	98269,74	101760,73	30
	Secunda	Prima	Secunda	Prima	Secunda	Prima	
	47		46		45		

91633,12

TABVLA LINEARVM TANGENTIVM, QVAE REGIOMONTANO, MAVROLICO, REINOLDO, ET VIETEO dicitur Fœcunda.

Rhetico canon basis, & perpendiculi secundus. Bressio tabula Adscriptarum.

Cuius vsus est pro explorando vno ex lateribus circa rectum; quando scilicet reliquum latus circa rectum assumitur, vt sinus totus.

X

DE PLANIS

Vt recta G C est sinus arcus B C, vel recta C H est sinus arcus C D, quæ perpendicularis est ad diametrum F D. Sic etiam B G est sinus arcus B C: ducitur enim perpēdicularis ad subtensam A G C dupli arcus A B C.

6. Sinus rectus est dimidia subtensa dupli arcus, cuius dicitur sinus.

Vt linea G C, quæ est dimidia pars subtensæ A C, dicitur sinus rectus arcus B C, qui est semissis arcus A B C subtensi à recta A G C.

VEL

Sinus rectus est perpendicularis ab vno termino arcus illius, cuius dicitur sinus, ad circuli diametrum ductam ad alterum eiusdem arcus terminum. Sinus rectus appellatur etiam sinus primus.

Vt in antecedente schemate recta G C est sinus arcus B C, quæ ducta est ab vno termino ipsius arcus (nempe C) perpendiculariter ad diametrum B E ductam ad alterum terminum B illius arcus B C.

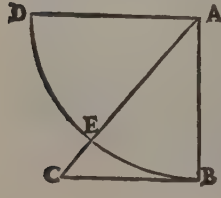
7. Sinus versus est portio diametri inter arcum, & sinum rectum.

Vt est linea B G, quæ est portio diametri inter arcum B C, ad quem refertur, & sinum rectum G C. Dicitur vero hæc linea G B sinus versus, quia verso modo disponitur respectu sinus recti. Appellari etiam consueuit sagitta, quoniā in modum sagittæ sita est inter arcum, & chordam.

8. Sinus rectus secundus est sinus complementi alicuius arcus.

Vt recta C H est sinus secundus rectus arcus B C, quia est sinus arcus C D, qui est complementum arcus B C.

9. Sinus integer, vel totus est semidiameter circuli, nempe sinus vel rectus, vel versus quadrantis circuli.

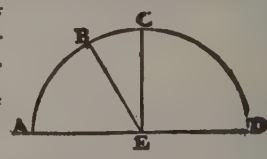


10. Amplitudo, seu magnitudo anguli rectilinei est arcus circuli super centrum anguli descripti inter ambo latera comprehensus.

Vt anguli B A C magnitudo est arcus E B circuli descripti facto centro in ipso angulo ad quod uis interualium: qui quidem arcus intercipitur inter ambo latera angulum concludentia. Quot igitur gradus complectitur ille arcus, tot graduum

duum dicetur etiam angulus, qualium vltus rectus est 90. gradus.

11. Cōplemētū anguli est ipsius à recto differentia, siue angulus sit maior recto, siue minor.



Vt in semicirculo A C D angulus rectus est C E A: quare complemētū anguli B E A, vel anguli B E D est angulus C E B, nempe differentia vniuscuiusq; illorum à recto.

12. Tangens alicuius anguli dicitur illud latus in angulo recto, quod perpheriam tangit, quæ describitur facto cētro in extremo reliqui lateris circa rectum ad eius interuallum, & refertur ad angulum in centro.

Vt in figura 10. def. linea C B est tāgens anguli C A B: quia facto centro in A extremo reliqui lateris A B circa rectum, & ad interuallū A B descripto circulo D E B, linea C B ipsum tangit; cum sit perpendicularis ad extremum diametri A B per coroll. 16. tertij elem. Has lineas tangentes Bressius Adscriptas appellat, & earum tabulam Adscriptarum, alij tabulā Fęcundam, vt Regiomontanus, Maurolicus, & Reinoldus.

13. Secans alicuius anguli dicitur latus recto angulo oppositum, quādo in altero acutorum angulorum descriptus fuerit circulus ad interuallum lateris circa rectū: hæc enim linea secare videtur huiusmodi circulum, & refertur hæc secans ad angulum in centro.

Vt in eademmet figura latus A C, quod angulo recto B opponitur, est secans anguli B A C; quia intersecat circulum D E B, qui descriptus est facto centro in ipso angulo B A C. Has secātes Bressius Hypothenusas vocat, & earum tabulā Hypothenusarum: alij tabulam Bentficā, vt Maurolicus.

14. Tangens alicuius arcus dicitur recta tāgens extremitatē illius arcus, quæ concurrit cū semidiametro circuli producta per alterū extremum eiusdem arcus vltra circulum.

Vt in iam dicta figura linea C B dicitur tangens arcus E B, quia tangit eum in extremitate B, & terminata est à semidiametro A E transeunte per extremum arcum E producta extra circulum vsque ad C.

15. Secans alicuius arcus est recta subtendens angulum rectum,

qua

Plate 19. G A. Magini. De planis triangulis, Liber unicus, 1592.

libri quatuor published at Leyden in 1591,[118] the tables of sines, tangents, and secants, and the method employed in their calculation have a great similarity to those of Vieta and Finck. The plane trigonometry of Lansberg shows no originality in content, but his arrangement of definitions and propositions is far superior to that of Vieta and Clavius, in so far as it is less complicated and more systematic.

The first book is devoted to the definitions of the trigonometric functions. Lansberg, following Bressieu, uses the term "radius" in place of the "sinus totus." For purposes of illustration, Item 21 under the caption "Definitions" will be given: "The secant of an arc is equal to the tangent of the arc and the tangent of half its complement."

The given arc is DE, the secant is AG, the tangent of DE is GE; the complement of DE is BD; half the complement is CB which is equal to FE; HE is the tangent of FE. I say that the secant AG is equal to EG, tangent of the given arc, and EH, tangent of half its complement.

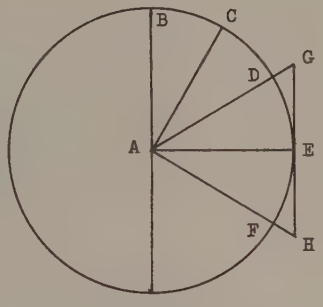


Fig. 120

Angle GAH is equal to angle CAE, angle EAH is equal to angle BAC, hence angle EHA or GHA is the complement of BAC and hence equal to CAE and therefore equal to GAH. So triangle GAH has angle A equal to angle H, therefore GA equals GH.

Porism:

When the tangent of an arc and the tangent of half its complement are added, the result is the secant of the given arc. For if EG, the tangent of the arc ED, is added to EH, the tangent of half the complement which is EF, the sum is HG which is equal to AG, the secant of the given arc DE.

The second book contains the method of constructing the tables and the tables themselves. Lansberg's only references here to other writers are in connection with the division of a circle into 360 degrees: "Haec circuli divisio est Ptolemaei recentiorum Mathematicorum..." and the division of the diameter into parts: "Ptolemaeus diametrum assumit particularum 120: Arzahel 300. Neoterici [modern authors], 20000000..."

The third book is devoted to the solution of plane triangles. Numerical illustrations accompany each of the fifteen demonstrations. Item 14 gives the statement and proof of the Sine Law, which differs very little from that given by Regiomontanus. Like Finck and Clavius, Lansberg uses the Tangent Law for the solution of a triangle when two sides and the included angle are given. His proof and figure for this law are exactly the same as those of Finck.[119]

Book IV treats of spherical trigonometry. The first eleven items concern spherical geometry. In the solution of spherical triangles Lansberg employs a device similar to that of Bressieu,[120] namely, the marking of the given parts of a triangle by two strokes.

In Item 17 of this book Lansberg produces a new proof for the Cosine Theorem. This is the first time that the theorem appears in print for angles as well as sides. There are sufficient evidences that the theorem for angles was known to Vieta and also to Tycho Brahe.[121] Lansberg's proof for sides, as given below, was copied verbatim by Stevin.

In an oblique-angled triangle the square of the radius is to the product of the sines of the two sides, as the vers sine of the angle included by these two sides is to the difference of the vers sine of the third side and the vers sine of the difference of the two sides.

118. The edition published at Leyden in 1631 was used. The 1591 edition is at the John Crerar Library.
119. See p. 89.
120. See p. 88.
121. A. von Braunmühl, op. cit., p. 192.

The square of the radius is to the product of the sines of the two angles as the vers sine of the side included by the two angles is to the difference of the vers sine of the third angle and the vers sine of the difference of the two angles from a semicircle.

In modern notation the theorems are:

$$\frac{1}{\sin a \sin b} = \frac{\text{vers sin } C}{\text{vers sin } c - \text{vers sin } (b - a)}$$

$$\frac{1}{\sin A \sin B} = \frac{\text{vers sin } c}{\text{vers sin } C - \text{vers sin } 180^{\circ} - (A - B)}$$

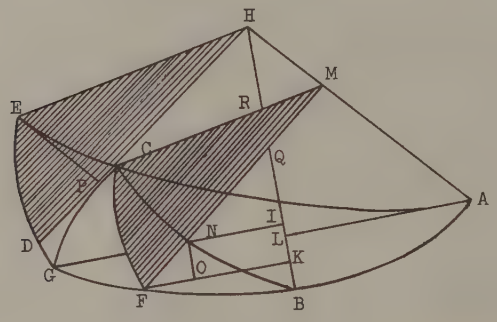


Fig. 121

ABC is the given triangle.

AB and AC are produced to D and E so that AE and AD are quadrants of great circles.

CF is constructed so that CA = FA.

With B as a pole CG is described so that BG = BC.

Hence, BF = AF - AB = AC - AB and GF = BG - BF = BC - (AC - AB).

H is the center of the sphere; HE, HA, and HB are radii of the sphere.

The perpendiculars AI, FK, and GL to the radius HB are the sines of the arcs AB, BF, and BG respectively; therefore the respective vers sines of these same arcs are BI, BK, and BL.

Observe that KL = BL - BK = vers sin BG - vers sin BF; consequently KL = vers sin BC - vers sin (AC - AB).

FM is perpendicular to HA and hence is the sine of AF.

GL and FM intersect at N; in the plane of ADH make NO parallel to HB, then NO = LK = vers sin BC - vers sin (AC - AB).

EP is perpendicular to HD and is the sine of DE; DP is the vers sine of DE.

Observe that FC and GC are perpendicular to the quadrant ABD, because ABD passes through the poles, A and B, of FC and GL respectively.

CN is the intersection of the two planes CGL and CFM which are perpendicular to the plane of the quadrant HAD and hence are perpendicular to the plane of HAD also. But CN passes through the point of intersection of the two lines GL and FM which lie in the plane HAD, hence CN is perpendicular to GL and FM.

Because CN is perpendicular to FM it is the sine of the arc FC; the vers sine of FC is then FN.

L. VALENTINI OTHONIS PARTHENOPOLITANI.

METEOROSCOPIVM NVMERORVM PRIMVM.

MONSTRANS PROPORTIONEM SINGVLORVM PARALLELORVM AD ÆQVATOREM VEL MERIDIANVM.

Anno Salutis.
CIↃ IↃ XCVI.

Plate 20. L. Valentin Otho, Extensive tables including all six functions, 1596. These tables were projected by Rheticus.

ANGVLO RECTO IN PLANITIE PARTIVM 10000000000. PONITVR.

ANGVLVM RECTVM		MINVS LATVS INCLVDENT. ANG: RECTVM.				
Perpendiculum	Different.	Hypotenuſa	Different.	Baſis	Different.	°
90420565 4	488779	11104549210 8	59572048	11059430961 0	59814944	50
90469443 3	488784	11098598415 6	59507952	11053455876 5	59750845	1 0
90518321 7	488788	11092654019 4	59443962	11047487189 2	59686873	1 0
90567200 5	488793	11086716024 3	59379951	11041524903 5	59622837	1 0
90616079 8	488796	11080784339 4	59316289	11035568984 3	59559192	2 0
90664959 4	488802	11074859147 0	59252484	11029619446 0	59495383	1 0
90713839 6	488804	11068940256 6	59188504	11023676260 5	59431810	49
90762720 0	488810	11063027713 9	59125427	11017739435 5	59368329	1 0
90811601 0	488814	11057121508 7	59061032	11011808936 1	59304960	1 0
90860482 4	488819	11051221630 9	58998778	11005884767 7	59241684	1 0
90909363 4	488822	11045328070 2	58935607	1999996691 70	59178507	2 0
90958246 5	488828	11039440828 8	58872424	10994055538 47	59115323	1 0
91007129 3	488830	11033559872 1	58809567	10988150378 6	59051471	48
91056012 3	488835	11027685214 3	58746578	10982251188 7	58989489	1 0
91104895 8	488840	11021816833 3	58683810	10976358517 0	58926719	1 0
91153779 8	488844	11015954718 9	58621144	10970472112 3	58864047	1 0
91202564 2	488849	11010098861 2	58558577	10964591963 5	58801488	2 0
91251549 1	488854	11004249250 1	58496111	10958718061 8	58739017	1 0
91300434 5	488856	10998405877 6	58433745	10952850396 0	58676638	47
91349320 1	488862	10992568739 9	58371357	10946988956 3	58614267	1 0
91398206 3	488866	10986737880 8	58309311	10941133747 6	58552217	1 0
91447092 9	488870	10980913096 5	58247123	10935284744 1	58490035	1 0
91495979 9	488874	10975094581 1	58185154	10929441937 8	58428063	2 0
91544867 3	488879	10969282252 7	58123284	10923605311 8	58366200	1 0
91593755 4	488883	10963476101 3	58061514	10917774875 3	58304425	46
91642643 5	488888	10957676129 3	57999720	10911950611 4	58242639	1 0
91691532 3	488891	10951882302 7	57938266	10906132493 3	58181181	1 0
91740421 4	488897	10946094637 7	57876670	10900320535 2	58119581	1 0
91789311 1	488901	10940313094 6	57815411	10894514702 3	58058329	2 0
91838201 2	488905	10934537693 6	57754010	10888715009 9	57996924	1 0
91887091 7	488906	10928768410 9	57692827	10882921436 2	57935737	45
91935982 5	488914	10923005124 88	57631621	10877133982 3	57874539	1 0
91984873 9	488917	10917248117 37	57570751	10871352614 7	57813676	1 0
92033765 6	488924	10911497199 7	57509740	10865577734 98	57752649	1 0
92082658 0	488926	10905752293 4	57449063	10859808150 8	57691990	2 0
92131550 6	488932	10900013469 0	57388244	10854045035 1	57631157	1 0
92180443 8	488935	10894280704 9	57327641	10848287579 0	57570461	44
92229337 3	488939	10888553991 5	57267134	10842536974 1	57510049	1 0
92278231 2	488944	10882833331 2	57206603	10836792021 4	57449127	1 0
92327125 6	488949	10877118698 5	57146407	10831053088 9	57389325	1 0
92376020 5	488953	10871410083 8	57086067	10825320189 5	57328993	2 0
92424915 8	488958	10865707447 8	57026060	10819593291 4	57268982	1 0
92473811 6	488961	10860010886 7	56961611	10813872408 4	57208830	43
92522707 7	488966	10854320300 9	56905858	10808157753 01	57148783	1 0
92571604 3	488971	10848635687 4	56846135	10802448624 5	57089056	1 0
92620501 4	488974	10842957048 6	56786388	10756745692 8	57029317	1 0
92669398 8	488979	10837284386 8	56726618	10791048733 86	56969542	2 0
92718296 7	488985	10831617680 9	56667059	10785357740 5	56909981	1 0
92767195 2	488987	10825956921 4	56607595	10779672668 3	56850522	42
92816093 9	488993	10820302099 0	56548224	10773993573 5	56791148	1 0
92864993 2	488996	10814653220 44	56488946	10768320385 9	56731876	1 0
92913892 8	489002	10809010240 0	56429644	10762653128 7	56672672	2 0
92962793 0	489006	10803373170 8	56370670	10756991768 4	56613603	1 0
93011693 6	489010	10797744201 76	56311554	10751336320 9	56554475	41
93060594 6	489015	10792116752 8	56252648	10745686762 4	56495585	1 0
93109496 1	489018	10786497369 3	56193835	10740043086 7	56436757	1 0
93158397 9	489024	10780883196 77	56134996	10734405294 0	56377527	40
93207300 3	489027	10775276821 3	56076484	10728773352 0	56319420	1 0
93256203 0	489032	10769676443 83	56017830	10723147275 8	56260762	2 0
93305106 2	489037	10764085000	55959383	10717527044 5	56202313	1 0
Baſis	Different.	Hypotenuſa	Different.	Perpendiculum	Differ nt.	40

SERIES. TERTIA SERIES.

84

F 3

Plate 21. L. Valentin Otho, Extensive tables including all six functions, 1596. These tables were projected by Rheticus.

Recalling how Fig. 121 was constructed, it is to be observed that the right section MFC is similar to the right section HDE and hence the following proportion is true:

$$\frac{\text{DH or radius}}{\text{FM or sin AC}} = \frac{\text{DP or vers sin A}}{\text{FN or verse sin FC}}.$$

Observe that triangle FON is similar to triangle FKQ which is similar to triangle HMQ which is similar to HAI, hence the following proportion is true:

$$\frac{\text{HA or radius}}{\text{AI or sin AB}} = \frac{\text{FN or vers sin FC}}{\text{NO or vers sin BC - vers sin (AC - AB)}}$$

The multiplication of the two proportions gives the formula which expresses the Cosine Theorem for sides. Lansberg did not prove this theorem for angles.

Braunmühl[122] has asserted that Lansberg's knowledge of the Cosine Theorem for angles might indicate that he had discovered Vieta's method for the construction of his reciprocal figures. However, Lansberg does not advance the theory of polars beyond what Vieta has done. Briefly, this is what Lansberg does:

ABC is the given triangle, B is an oblique angle, A and C are acute. F is the pole of AB on the circle AHKA: G is the pole of AC on the semicircle ACL, E is the pole of BC on the semicircle BCD.

F, C, G determine the semicircle NFI, A, E, G determine the semicircle GPL, the semicircle passing through E and F having B as a pole is HEMK. Hence, another triangle, namely EFG is determined such that EF equals the difference between a semicircle and angle ABC, GF equals angle BAC, GE equals angle ACB. Angle EFG equals AB, angle FEG equals BC, and angle EGF is the supplement of side AC. In Fig. 122, which is the same as the one in Lansberg's text, triangle AGC is a quadrantal triangle.

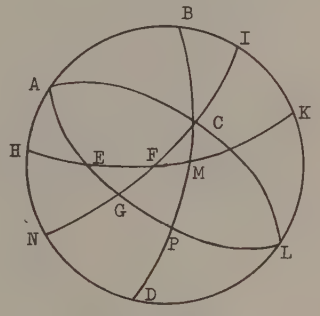


Fig. 122

Lansberg's trigonometry is on a par with that of his contemporaries. Compared with the treatises of Vieta and Clavius the Triangulorum geometriae has a more simplified exposition and a better systematization of ideas.

9. GIOVANNI MAGINI

The three trigonometric treatises of Magini are De planis triangulis...published at Venice in 1592, which is devoted entirely to plane trigonometry; Tabulae primi mobilis...compendium calculi triangulorum, published at Venice in 1604 and Bologna in 1609, which contains eighteen tables, mostly for astronomical calculations, with directions for their use and thirty-three propositions concerning the solution of right and oblique spherical triangles; and the Primum mobile...trigonometria sphaericorum, published in 1604 at Venice, in 1609 at Bologna[123] and Venice, and in 1615 at Frankfort and Bologna. This work consists of twelve books, the first four of which are devoted to spherical trigonometry.

At various places in the De planis triangulis Magini refers to Ptolemy, Copernicus, Rheticus, Maurolyco, Vieta, Bressieu, Finck and Clavius. The influence that these authors might have had on him is not evidenced by his treatise on plane triangles, for the De planis triangulis lacks some of the important items that occur in the works of several of the above mentioned writers. The credit given Magini for advancing trigonometry results from his treatment of spherical trigonometry and not from the De planis triangulis.

122. Ibid., pp. 192-193.
123. This edition is at U.M.

Magini begins his treatise on plane trigonometry by defining and illustrating arc, chord, sine, etc. As a general rule he mentions the nomenclature of others. Thus, in his definition of "tangent" he says:

> Bressieu called these lines 'adscripta' and their tables 'tabula adscriptae.' Others called them 'tabula fecunda' as Maurolyco, Regiomontanus and Rheinoldus.

In his definitions Magini uses the expression "sinus integer" for the radius, however, he does not use it elsewhere in the text. The co-functions of the sine, tangent, and secant are designated second sine, second tangent, and second secant respectively.

Besides the tables of the trigonometric functions, Magini also has what he terms a "tabula tetragonica." This table offers a convenient way of finding the square root of a number or an approximation to the square root to at least eight places of accuracy. The square of a number may also be read from this table. The following illustration is taken from the beginning of the table. The largest square read from this table is $10100^2 = 102010000$.

Radices Centenariorum							
		100	200	300	400	500	600
Numeri Quadrati							
0	0	10000	40000	90000	160000	250000	360000
1	1	10201	40401	90601	160801	251001	361201
2	4	10404	40804	91204	161604	252004	362404
3	9	10609	41209	91809	162409	253009	363609

For practice in using the trigonometric tables, Magini proposes problems such as: "Given an arc or an angle to find the sine or cosine."

Magini divides the plane trigonometry proper into three parts; the first considers some general propositions of plane triangles; the second, the solution of right triangles; and the third, the solution of oblique triangles.

Some of the propositions of the first section have occurred many times in previous texts, such as:

> If from an angle of a scalene triangle with known sides a perpendicular is drawn, investigate the lengths of the segments made on the base by perpendicular.

The Sine Law, as first stated and proved by Al-Biruni, with its accompanying figure is given as Proposition X.

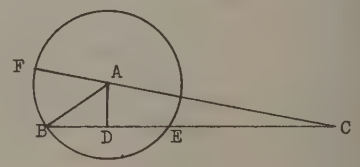


Fig. 123

In the second section Magini states only two propositions for the solution of right triangles, but in so doing he excludes none of the cases for all of them are considered. His proofs are usually given according to two methods, one which employs sines and cosines, and the other which uses tangents and secants. Numerical illustrations accompany each proof.

Magini follows the same plan for the solution of oblique triangles. He does not consider the solution of a triangle when two sides and the included angle are given.

Published with Magini's plane trigonometry is his treatise on practical geometry dealing with the measurements of distances, heights, and depths which are ascertained by

BARTHOLO-

mæi Pitiſci Grunbergenſis

TRIGONOMETRIÆ

Liber Qvartvs.

De dimenſione Triangulorum Sphæricorum.

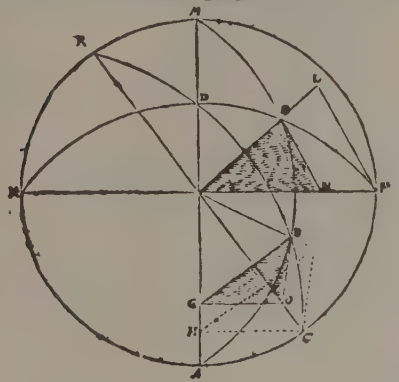
Axiomata proportionum in Triangulis Sphæricis exiſtentium præcipua, & ad omnem eorum ſolutionem abunde ſufficientia ſunt quatuor.

AXIOMA PRIMVM.

In Triangulis Sphæricis rectangulis pluribus, acutum ad baſes eundem habentibus:

Sinus hypotenuſarum & perpendiculorum omnes

Schema LVII.



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BARTHOLO

mæi Pitiſci Grunbergenſis

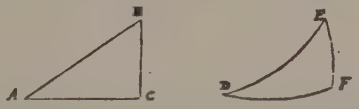
TRIGONOMETRIÆ

Liber Primvs.

De generibus & affectionibus Triangulorum.

I. Trigonometria eſt doctrina de dimenſione Triangulorum.

II. Triangulum eſt figura tribus lateribus tres angulos comprehendens: *Vt ſunt figura A B C. & D E F.*



Schema I.

III. Latera duo quælibet ſunt crura anguli à ſe comprehenſi; tertium, baſis. *Vt latera A B. & A C. ſunt crura anguli B A C. latus B C. eſt eiusdem anguli baſis.*

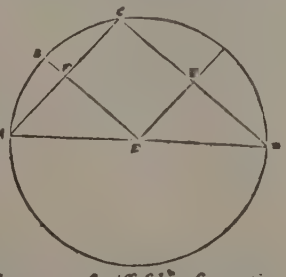
IV. Latus unumquodq; dicitur ſubtendere angulum ſibi oppoſitum. *Vt latus A B. ſubtendit angulum A C B. Latus A C. ſubtendit angulum A B C. Latus B C. ſubtendit angulum B A C.*

V. Latera maiora maiores angulos ſubtendunt. Subintellige: Et minora minores, & æqualia æquales. *Veritas Theorematis per ſe manifeſta eſt. Demonſtratur tamẽ apud Euclidem ad 18. & 19. p. 1. & apud Regiomontanum ad 42. & 43. prop. 3. Luculenter etiam confirmabitur infrà per*

B quartum

mè reperitur per 34. p. 1. hoc eſt per figuram quadrilateram circulo inſcriptam. *In qua inſcriptione etſi mera ſunt ſubtenſæ, non ſinus: tamen, quia ſinus ſunt ſemiſſes ſubtenſarum: eadem autem eſt proportio totius ad totum, quæ dimidij ad dimidium: Ideo ſi ſubtenſis datis numeros ſemiſſium ſubtenſarum ſiue ſinuum attribuas, etiam ſubtenſæ inuentæ ſinus erunt. Hoc eſt, ut exemplo perſpicuo rem per ſe non nihil obſcuram illuſtremus:*

Schema XXXIV.



Si ſubtenſam A D *habeas pro ſinu* A E, *ſubtenſa* A C *erit ſinus* A F *& ſubtenſa* C D *erit ſinus* C G *vel* F E. *Namut* A D *ad* A E *ita* A C *ad* A F *& ita* C D *ad* C G. *adeoq; ut* A D *eſt* A E *ſic* A C *eſt* A F. *& ſic* C D *eſt* C G. *per 46. p. 1.*

Quod ſi ſubtenſa A C *habeatur pro ſemiſſe ſubtenſæ* A F *etiam arcus* A C. *habebitur pro arcu* A B. *adeoq; totus circulus* A B C D. *ſecundum hanc hypotheſin non erit niſi partium 180.*

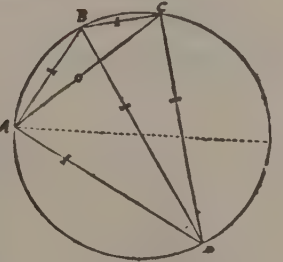
Datis ſinibus duorum arcuum inæqualium coniunctim quadrante minorum, unà cum ſinibus complementorum, ſinum ſummæ duorum illorum arcuum inuenire.

Sint ergo dati duo arcus inæquales A B *20. grad.*
B C. *15. grad. eorumq; complementa* A D *70. grad.*
C D. *75. gr. Et horum omnium ſinus.*

A B. 34202.

A B. *34202* X A D. *93969.*
B C. *25881.* X C D. *96592.*
Vna cum radio B D. *100000.*

Schema XXXV.



Quæratur autem ſinus A C *ſummæ duorum arcuum* A B *&* B C, *35. gr.*

Primùm multiplicentur alternatim ſinus arcus unius per ſinus complementi alterius nempe A B *per* C D *&* B C *per* A D. *Tum producta componantur. Denig; ſumma per radium* B D *diuidatur, hoc eſt, nota quinq; á dextris abſcindantur. Nota reliqua erunt ſinus ſummæ dictorum duorum arcuum* A B *&* B C, *nempe ſinus arcus* A C. *quæſitus, hoc modo:*

Productum primum 3303659584.
Productum ſecundum 2432011689.
Summa ——— 57356 | 71273.
Sinus
A C. *35. gr.*

Datis Sinibus duorum arcuum inæqualium, coniunctim quadrante

Vt AB *5. ad* ACB. *100000. ita* BC. *3. ad* BAC. *60000.*
Ergo angulus BAC. *est 36. gr. 53. min.*

NOTA. In usu huius axiomatis: Si dentur duo latera, unà cum angulo, minori datorum laterum opposito: & quæratur angulus, maiori datorum laterum oppositus: quia is angulus tam acutus quam obtusus esse potest: & sinus utriuslibet est idem per 1. c. 7. p. 2. Ideo ne in calculo erres: qualis sit is angulus quem quæris, acutus an obtusus, ante omnia constitutum habeas oportet. *Verbi gratia. Si in Triangulo* EIF *dentur latera* EI *&* EF. *unà cum angulo* EFI. *quæratur autem angulus* EIF. *Si per solos numeros ista data sint: Triangulum autem ipsum appictum non sit: quia eadem data etiam in Triangulo* DEF. *inesse possunt: propterea quod latus* ED *potest esse æquale lateri* EI. *angulo* EFI. *& latere* EF *iisdem manentibus: ideo ne pro obtuso* EIF. *inuenisse te putes acutum* EDF. *ante omnia constitutum habeas oportet, qualis sit angulus lateri* EF *oppositus, quem quæris: acutus nimirum an obtusus?*

Schema XLIX.

AXIOMA QVINTVM.
In Triangulis planis uniuersis:

Vt summa duorum laterum ad differentiam eorundem: ita tangens dimidij summæ duorum angulorum oppositorum, ad tangentem differentiæ infra uel supra dimidium.

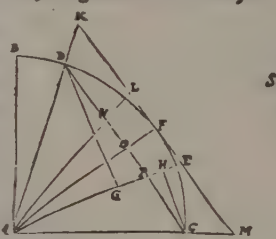
DECLARATIO

DECLARATIO. In Triangulo plano obliquangulo ABC. *dico tangentem dimidij summæ duorum angulorũ ad* A *&* B. *esse ad tangentem differentiæ anguli* B. *supra, & anguli* A. *infra dimidium: Vt est summa duorum laterum* BC *&* AC. *oppositorum ad differentiam eorundem.*

Schema L.

DEMONSTRATIO. Descripto enim quadrante ABC *statuantur anguli* DAE *&* EAC. *angulis* ABC *&* BAC. *æquales. Ac proinde, sit summa duorum illorum angulorum, angulus* DAC. *Dimidium illius summæ sit* DAF *uel* FAC. *Differentia anguli* DAE. *supra dimidium* DAF *uel anguli* EAC. *infra dimidium* FAC. *sit angulus* FAE. *Subtensa summæ duorum angulorũ sit recta* DC. *Sinus anguli maioris* DAE *sit recta* DG. *Sinus anguli minoris* EAC *sit recta* CH. *Tangens dimidij summæ duorum angulorum sit* FM *uel* FK. *Tangens differentiæ infra uel supra dimidium sit* FE. *Iam Triangula* GDP *&* CPH. *sunt æquiangula per 4. c. 49. p. 1. Propter æquales* DPG *&* CPH. *per 13. p. 1. & rectos ad* G *&* H. *per 3. c. 7. p. 2. Ergo in hoc secundo schemate: Vt* PD *ad* GD. *ita* PC *ad* CH. *per 46. p. 1. Sicut in primo:*

Schema LI.



K *Vt* AC

826 PROBLEMATVM ALTIMETRICORVM

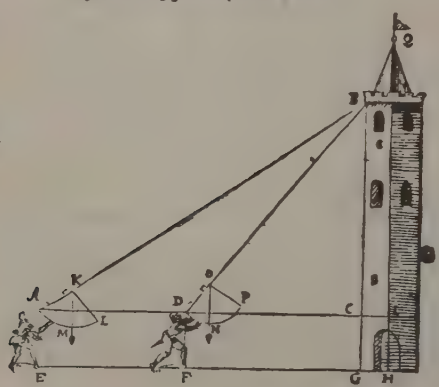
(ita distantia AC.
200. ped.
ad alt. BC. $113\frac{824}{1000}$. id est ferè 114. ped.

NOTA. Si metiri libeat altitudinem globi Q. *non danda erit distantia* AC. *sed* AI. *quia punctum* Q. *puncto* C. *non est perpendiculare.*

PROBLEMA SECVNDVM.

Data distantia, hypotenusam inuenire.

Schema LXXXIII



Rursum sit data distantia AC. 200. pedum, quæratur autem hypotenusa AB. quot pedum?
Obseruatis & repertis per Quadrantem angulis, ut ante, Dico: Vt ABC

TABVLÆ

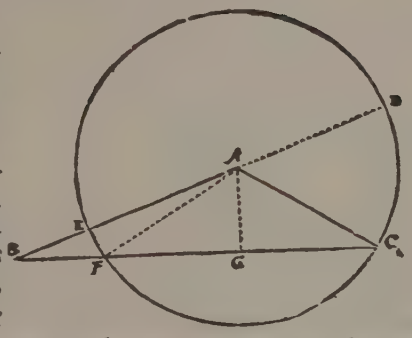
52	Sinuum		Tangentium		Secantium		89
31	53754	84324	63748	156868	118591	186031	29
32	53779	84308	63789	156767	118613	185946	28
33	53804	84292	63830	156667	118635	185861	27
34	53828	84277	63871	156566	118657	185777	26
35	53853	84261	63912	156466	118679	185692	25
36	53877	84245	63953	156366	118701	185608	24
37	53902	84230	63994	156265	118723	185523	23
38	53926	84214	64035	156165	118745	185439	22
39	53951	84198	64076	156065	118767	185355	21
40	53975	84182	64117	155966	118790	185271	20
41	53999	84167	64158	155865	118812	185187	19
42	54024	84151	64199	155766	118834	185103	18
43	54049	84125	64240	155666	118856	185019	17
44	54073	84120	64281	155567	118878	184935	16
45	54091	84104	64322	155467	118901	184852	15
46	54122	84188	64363	155368	118923	184768	14
47	54146	84072	64404	155269	118945	184685	13
48	54171	84057	64446	155170	118967	184601	12
49	54195	84041	64487	155071	118990	184518	11
50	54220	84025	64528	154972	119012	184435	10
51	54244	84009	64569	154873	119034	184352	9
52	54269	83994	64610	154774	119057	184260	8
53	54293	83978	64652	154675	119079	184186	7
54	54317	83962	64693	154576	119102	184103	6
55	54342	83946	64734	154478	119124	184020	5
56	54366	83930	64775	154379	119146	183938	4
57	54391	83915	64817	154281	119169	183855	3
58	54415	83899	64858	154183	119191	183773	2
59	54439	83883	64899	154085	119214	183690	1
60	55464	83867	64941	153987	119236	183608	0

Plate 23. Bartholomew Pitiscus, Trigonometriae, 1600.

DECLARATIO.

Sit Triangulum obliquangulum ABC *eiusq; latus minimum* AC. *maximum* BC. *Radio lateri minimi* AC. *centro* A *describatur circulus* CDEF *secans reliqua duo latera in punctis* E *&* F. *Et latus* AB. *porro producatur in* D. *Eritq; tota* BD *summa laterũ* AB *&* AC. *nam* AC *&* AD *aquantur per structuram.* BE *uerò erit differentia laterũ* AB *&* AC. *Nam* AE *&* AC *rursus aquantur per structuram. Dico.*

Schema LII.



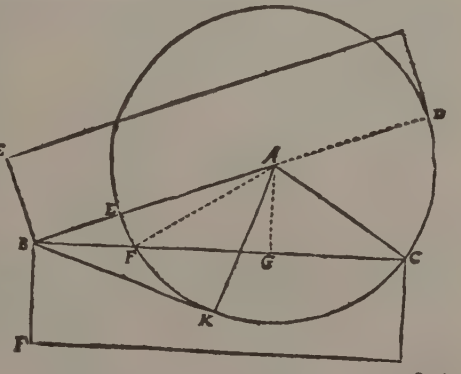
Primum esse, ut CB *ad* BD. *ita* EB *ad* BF.

Deinde, perpendiculum AG. *bisecare rectam* FC.

DEMONSTRATIO.

Nam, quod ad primum attinet, Rectangula aqualia habẽt latera reciprocè proportionalia per 2. c. 42. p. 1. Atqui oblon-

Schema LIII.



ga. facta

Plate 24. Bartholomew Pitiscus, Trigonometriae, 1600.

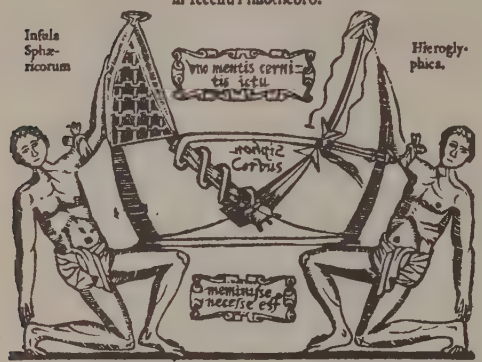
DICLIDES COELOMETRICÆ
seu

VALVÆ ASTRONOMICÆ
VNIVERSALES

Omnia artis totius munera Psephophoretica in sat modicis finibus duarum Tabularum Methodo noua, generali, & facilima continentes.

Praeunte Directionis accurate consumatâ Doctrina, Astrologis hactenus plurimùm desiderata.

Authore NATHALE TORPORLAEO Salopiensi in secessu Philotheoro.

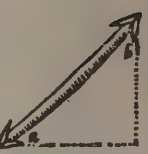


Fragilis & laboriosa mortalitas in partes ista digessit, vt portionibus quisque coleret, quo maximè indigeret; PLIN.

Sed quæ carptim singula comflant
Eadem nos hic missa ingamus. HORT.

LONDINI
Excudebat FELIX KINGSTON. 1602.

cuius forma hæc est: Neque equidem est adeo longè petita similitudo, vt quæ quibus respondeat partibus opus sit exponere; termini sunt A, B, A B.



Forfex. Ergo tertia Triplicitas inter plurilaterales prima, vbi coeunt Hypothenusa latus, & angulus incluſus; dicetur *Forfex*, ob coitionem duarum linearum, vimque sectricem in puncto coitionis; quibus combinationis istius termini tres, eorumque inter sese positura dilucidè exprimuntur; vt videtur hac figura per AB: CB: & angulum ABC: quo semper suam sortitur actionem ipsa *Forfex*. Sed si cui proprietas non rectè seruata videatur, propter æqualitatem laterum *Forficis*, hanc tum forte rectius direxit vncinum, quali vtuntur pueri, ad puces sublimiores prehendendas, aut pedum, quo pastores otium pedes irretiunt; in vtroque enim est inæqualitas laterum, & virtus agendi in eorum concursu manet. Verum satis dilucida est *Forficis* similitudo; namque ista cõcinnitas nequaquam est adeò minutim affectanda, modo perspicua tenetur applicatio; & si bene animaduertas, quantumcunque inæqualia videantur huius *Forficis* hiantia latera; in compressu tamen, quanto propius accedant, tanto minor appareb. inæqualitas, quoad coniunctis nulla fuerit.

Siphon. Iam quarta, & inter plurilaterales sequens, ad quam conficiendã Hypothenusa lat. & angulus lateri oppositus conuenitit, aliquo modo assimilatur Tollenoni, machinæ scilicet illi bellicæ, qua solebant olim Milites ad murorum, quos obsidebant, fastigia sublimes ferri; aut virgæ piscatoriæ, quæ constat ex baculo, filo, & hamo vt angulo; sed aptissimè præ se fert speciẽ fistulæ illius retortæ, qua vtuntur quidam, ad exugendam aquam locis reconditioribus, cuius figura hæc est, & constat è parti-

bus

Quincunx vel Porta Sinistra — *Aperta Siphoni, Fundæ, Carceri.* — *Pars III*

Differentiæ Horizontales & Contingentia Asylions:

Et æquationes Argumentorum maximæ. Fere quæque in Sphæricis.

Plate 25. N. Torporley, Diclides Coelometricae, 1602.

trigonometric methods. The problems with their elaborate figures are somewhat similar to the illustration that has been selected from the Geometrica practica of Clavius.[124]

The essential trigonometric feature of both editions of the Tabulae primi mobilis are the four tables constructed in a unique manner for giving the solution of right and oblique spherical triangles. Collectively, the tables[125] are known as "tabula generalis," individually as "tabula primae analogiae seu tabula prima primi mobilis," etc. In the 1604 edition these tables are calculated for the interval of one degree; the 1609 edition has the calculations for an interval of ten minutes. The following is an illustration of the manner in which a triangle can be solved by these tables. This is the first of the thirty-three problems that Magini proposes: "Given the base of a right-angled spherical triangle and an angle adjacent to the base to determine the side opposite the given angle."

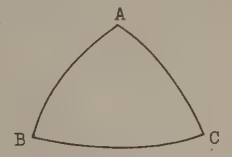


Fig. 124

ABC is the given triangle; B is a right angle; AC = 64°30'; angle C = 32°20'.

The implied formula is sin AB = sin AC sin C. The "Tabula primae analogiae" will be used because the formula exhibits a product of sines; the following is a portion of this table containing the part that is applicable to this problem.

	61	62	63	64	65	
	p '	p '	p '	p '	p '	
31	26 46	27 3	27 19	27 35	27 50	
32	27 37	27 54	28 10	28 27	28 42	
33	28 27	28 45	29 2	29 19	29 35	

Column 64 and row 32 produce 28° 27'; column 65 and row 32 produce 28° 42'; by means of proportional parts 64° 30' with 32° is estimated to be 28° 34'.

Column 64 and row 33 produce 29° 19'; column 65 and row 33 produce 29° 35'; by means of proportional parts 64° 30' with 33° is estimated to be 29° 28'.

By means of proportional parts 64° 30' with 32° 20' is estimated to be 28° 51' which is the length of the side AB in triangle ABC.

The 1609 edition of the Tabulae primi mobile contains Magini's "Magnus Canon trigonometricus" calculated for every minute of the quadrant for a radius equal to 10000000 and arranged for the direct reading of the co-functions.[126]

There is an interval of twelve years between the dates of publication of the plane and spherical trigonometry of Magini. The interim must have been well spent, for in contrast to the De planis triangulis, which was scarcely on a par with contemporary works, the Primum mobile exhibits Magini's complete mastery of spherical trigonometry.

The Primum mobile is being considered a little out of its chronological order. Other publications, not yet considered, influenced the spherical trigonometry of Magini. It is necessary also to recall that by 1604 Magini must have had access to the Opus palatinum de triangulis of Rheticus, which was published by Valentine Otho in 1596.

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124. See p. 94.

125. One of these tables, the third, occurs first in the work of Torporley, considered later in this chapter. Torporley's work on spherical trigonometry, with this table, was published previously to that of Magini. See p. 107.

126. A. von Braunmühl, op. cit., p. 234.

From Pitiscus[127] Magini took the term "trigonometry" as well as the idea of intermingling definitions pertaining to plane trigonometry with those of spherical tirgonometry.

The division of right-angled spherical triangles into six forms is taken from Rheticus,[128] however, Magini interchanges the second and the third forms. Magini also classifies the oblique spherical triangle under ten forms.

The first thirty-two theorems of Magini are the fundamental identities for a right-angled spherical triangle; they are listed here in modern notation.

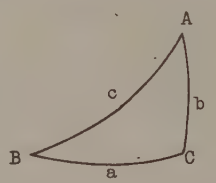

Fig. 125

(1) 1/sin A = sin c/sin a
(2) 1/cos a = cos b/cos c
(3) 1/sin A = cos b/cos B
(4) 1/sin A = csc a/csc c
(5) 1/sin c = csc a/csc A
(6) 1/csc A = sin a/sin c
(7) 1/cos c = sin a/sin A
(8) 1/cos a = sec c/sec b
(9) 1/sec a = cos c/csc b
(10) 1/sin A = sec B/sec b
(11) 1/cos b = sec B/csc A
(12) 1/csc A = cos B/cos b
(13) 1/sec a = cos A/sin B
(14) 1/scs A = csc c/csc a
(15) 1/sec a = sec b/sec c
(16) 1/csc A = sec a/sec b
(17) 1/tan A = sin b/tan a
(18) 1/cot a = cos B/cot b
(19) 1/cos c = tan A/cot B
(20) 1/sin a = cot b/cot B
(21) 1/cot A = tan a/sin b
(22) 1/csc a = tan b/tan B
(23) 1/cos A = tan c/tan b
(24) 1/tan a = cot c/cos B
(25) 1/sec A = cot c/cot b
(26) 1/sec c = cot A/tan B
(27) 1/cot A = cot B/cos c
(28) 1/csc a = cot B/cot b
(29) 1/cot a = tan A/csc b
(30) 1/sec A = tan b/tan c
(31) 1/tan c = cot a/sec B
(32) 1/tan A = tan B/sec c

In the section devoted to oblique spherical triangles Magini states the Cosine Theorem, which can be expressed in modern notation as:

$$\frac{1}{\sin b \sin c} = \frac{\text{vers sin A}}{\text{vers sin a} - \text{vers sin (b - c)}} .$$

Magini uses this theorem for solving an oblique triangle when two sides and the included angle are given, just as his contemporaries had done.

Magini's text contains many numerical illustrations. His treatment of polar triangles is a modification of Vieta's method and an imitation of that of Pitiscus.[129]

Magini's Primum mobile is truly monumental and in the field of spherical trigonometry is comparable to the work of Vieta.

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127. See p. 102.
128. See p. 62.
129. See p. 103.

IO. ANTONII MAGINI
PATAVINI
MATHEMATICARVM
IN ALMO BONONIENSI GYMNASIO
PROFESSORIS:

TABVLÆ PRIMI MOBILIS,
QVAS DIRECTIONVM VVLGO DICVNT,

Quibus non ſolùm Directiones, tam ſecundum viam rationalem, quàm iuxta Ptolemæi formam; præcipuè verò ope duarum generalium Poſitionum Tabularum, omninò abſoluuntur:

Sed & vniuerſa Primi Mobilis doctrina adiecto nouo Fundamento Radiationum Planetarum in Aequatore, plurimùm illuſtratur.

Subnectuntur præterea

Compendium Calculi Triangulorum Sphæricorum, tum Rectangulorum, tum Obliquangulorum, per ingreſſus laterales in Tabulas Quatuor Analogiarum,

Nouum Apologia Ephemeridum eiuſdem Auctoris contra Danidem Origanum.

Ad Auguſtiſſimum Imperatorem

RVDOLPHVM II

VENETIIS, Apud Damianum Zenarium MDCIIII.
Cum licentia Superiorum, & Priuilegijs.

Plate 26. G. A. Magini, Tabulae primi mobilis, with trigonometry, 1604.

Tabula Primæ Analogiæ, seu

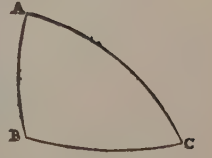
	61		62		63		64		65		66		67		68		69		70		71		72		73		74		75		
	P	′	P	′	P	′	P	′	P	′	P	′	P	′	P	′	P	′	P	′	P	′	P	′	P	′	P	′	P	′	
1	0	52	0	53	0	53	0	54	0	54	0	55	0	55	0	56	0	56	0	56	0	57	0	57	0	57	0	58	0	58	59
2	1	45	1	46	1	47	1	48	1	49	1	50	1	50	1	51	1	52	1	53	1	53	1	54	1	55	1	55	1	56	58
3	2	37	2	39	2	40	2	42	2	43	2	44	2	46	2	47	2	48	2	49	2	50	2	51	2	52	2	53	2	54	57
4	3	30	3	32	3	34	3	36	3	37	3	39	3	41	3	42	3	44	3	45	3	47	3	48	3	49	3	51	3	52	56
5	4	22	4	25	4	27	4	30	4	32	4	34	4	36	4	38	4	40	4	42	4	44	4	45	4	47	4	48	4	50	55
6	5	15	5	18	5	21	5	23	5	26	5	29	5	31	5	34	5	36	5	38	5	40	5	42	5	44	5	46	5	48	53
7	6	7	6	11	6	14	6	17	6	20	6	24	6	26	6	29	6	32	6	35	6	37	6	39	6	42	6	44	6	46	51
8	6	59	7	4	7	7	7	11	7	15	7	18	7	22	7	25	7	28	7	31	7	34	7	36	7	39	7	41	7	44	49
9	7	52	7	56	8	1	8	5	8	9	8	13	8	17	8	20	8	24	8	27	8	30	8	33	8	36	8	39	8	41	47
10	8	44	8	49	8	54	8	59	9	3	9	8	9	12	9	16	9	20	9	23	9	27	9	30	9	33	9	37	9	39	45
11	9	36	9	42	9	47	9	52	9	57	10	2	10	7	10	11	10	16	10	20	10	24	10	27	10	31	10	34	10	37	43
12	10	29	10	35	10	41	10	46	10	52	10	57	11	2	11	7	11	12	11	16	11	20	11	24	11	28	11	32	11	35	41
13	11	21	11	27	11	34	11	40	11	46	11	52	11	57	12	2	12	7	12	12	12	17	12	21	12	25	12	29	12	32	39
14	12	13	12	20	12	27	12	34	12	40	12	46	12	52	12	58	13	3	13	8	13	13	13	18	13	23	13	27	13	31	37
15	13	5	13	13	13	20	13	27	13	34	13	41	13	47	13	53	13	59	14	5	14	10	14	15	14	20	14	24	14	29	35
16	13	57	14	5	14	13	14	21	14	28	14	35	14	42	14	48	14	55	15	1	15	6	15	12	15	17	15	22	15	26	34
17	14	49	14	58	15	6	15	14	15	22	15	29	15	37	15	44	15	50	15	57	16	3	16	9	16	14	16	19	16	24	32
18	15	41	15	50	15	59	16	8	16	16	16	24	16	32	16	39	16	46	16	53	16	59	17	5	17	11	17	17	17	22	32
19	16	33	16	42	16	52	17	1	17	10	17	18	17	26	17	34	17	34	17	49	17	56	18	2	18	8	18	14	18	20	31
20	17	24	17	34	17	45	17	54	18	3	18	12	18	21	18	29	18	37	18	45	18	52	18	59	19	5	19	12	19	17	30
21	18	16	18	27	18	37	18	47	18	57	19	7	19	16	19	24	19	33	19	41	19	48	19	56	20	3	20	9	20	15	29
22	19	8	19	19	19	30	19	41	19	51	20	1	20	10	20	19	20	28	20	37	20	45	20	54	21	0	21	6	21	13	28
23	19	59	20	11	20	22	20	30	20	44	20	55	21	5	21	14	21	24	21	32	21	41	21	49	21	56	22	4	22	10	27
24	20	50	21	3	21	15	21	27	21	38	21	49	21	59	22	9	21	19	22	28	22	37	22	45	22	53	23	1	23	8	26
25	21	42	21	55	22	7	22	19	22	31	22	42	22	54	23	4	23	14	23	24	23	33	23	42	23	50	23	58	24	6	25
26	22	33	22	46	23	59	23	12	23	25	23	36	23	48	23	59	24	9	24	20	24	29	24	38	24	47	24	55	25	3	24
27	23	24	23	38	23	52	24	5	24	18	24	30	24	42	24	54	25	5	25	15	25	25	25	35	25	44	25	52	26	1	23
28	24	15	24	29	24	44	24	57	25	11	25	24	25	36	25	48	26	0	26	11	26	21	26	31	26	41	26	50	26	58	22
29	25	5	25	21	25	36	25	50	26	4	26	17	26	30	26	43	26	55	27	6	27	17	27	27	27	37	27	47	27	55	21
30	25	56	26	12	26	27	26	42	26	57	27	11	27	24	27	37	27	50	28	1	28	13	28	24	28	34	28	44	28	53	20
31	26	45	27	3	27	19	27	35	27	50	28	4	28	18	28	32	28	44	28	57	29	9	29	20	29	30	29	41	29	50	19
32	27	37	27	54	28	10	28	27	28	42	28	56	29	12	29	26	29	39	29	52	30	4	30	16	30	27	30	37	30	47	18
33	28	27	28	45	29	2	29	19	29	35	29	50	30	5	30	20	30	34	30	47	31	0	31	12	31	23	31	34	31	44	17
34	29	17	29	35	29	53	30	10	30	27	30	43	30	59	31	14	31	28	31	42	31	55	32	8	32	20	32	31	32	42	16
35	30	7	30	26	30	44	31	2	31	19	31	36	31	52	32	8	32	24	32	37	32	51	33	4	33	16	33	28	33	39	15
36	30	56	31	16	31	33	31	53	32	11	32	29	32	45	33	1	33	17	33	32	33	46	33	59	34	12	34	24	34	36	14
37	31	46	32	6	32	26	32	45	33	3	33	21	33	38	33	55	34	11	34	26	34	41	34	55	35	8	35	21	35	33	13
38	32	35	32	56	33	16	33	36	33	55	34	13	34	31	34	48	35	5	35	21	35	36	35	50	36	4	36	17	36	29	12
39	33	24	33	45	34	6	34	27	34	47	35	6	35	24	35	42	35	59	36	15	36	31	36	46	37	0	37	13	37	26	11
40	34	13	34	35	34	56	35	17	35	38	35	58	36	17	36	35	36	53	37	10	37	26	37	41	37	56	38	10	38	23	10
41	35	1	35	24	35	46	36	8	36	29	36	49	37	9	37	28	37	46	38	4	38	20	38	36	38	51	39	6	39	19	49
42	35	49	36	13	36	36	36	58	37	20	37	41	38	1	38	21	38	40	38	58	39	15	39	31	39	47	40	2	40	16	48
43	36	37	37	2	37	25	37	48	38	11	38	32	38	53	39	13	39	33	39	51	40	9	40	26	40	42	40	58	41	12	47
44	37	25	37	50	38	14	38	38	39	1	39	25	39	45	40	6	40	26	40	45	41	3	41	21	41	38	41	54	42	9	46
45	38	13	38	38	39	3	39	28	39	51	40	14	40	37	40	58	41	19	41	38	41	57	42	16	42	33	42	49	43	5	45
46	38	59	39	26	39	52	40	17	40	41	41	5	41	28	41	50	42	11	42	32	42	51	43	10	43	28	43	45	44	1	44
47	39	46	40	13	40	40	41	6	41	31	41	55	42	19	42	42	43	4	43	25	43	45	44	4	44	23	44	40	44	57	43
48	40	32	41	0	41	28	41	54	42	20	42	45	43	10	43	33	43	56	44	18	44	38	44	58	45	17	45	35	45	52	42
49	41	18	41	47	42	15	42	43	43	9	43	35	44	0	44	24	44	48	45	10	45	32	45	52	46	12	46	30	46	48	41
50	41	4	42	34	43	3	43	31	43	58	44	25	44	50	45	15	45	39	46	3	46	25	46	46	47	6	47	25	47	44	40
51	42	49	43	20	43	49	44	18	44	47	45	14	45	40	46	6	46	31	46	55	47	17	47	39	48	0	48	20	48	39	39
52	43	34	44	5	44	36	45	6	45	35	46	3	46	30	46	56	47	22	47	46	48	10	48	33	48	54	49	15	49	34	38
53	44	18	44	51	45	22	45	52	46	22	46	51	47	19	47	46	48	13	48	39	49	2	49	25	49	48	50	9	50	29	37
54	45	2	45	35	46	7	46	39	47	9	47	39	48	8	48	36	49	3	49	29	49	54	50	18	50	41	51	3	51	24	36
55	45	46	46	19	46	53	47	25	47	56	48	27	48	56	49	25	49	53	50	20	50	46	51	10	51	34	51	57	52	18	35
56	46	29	47	3	47	37	48	13	48	43	49	14	49	44	50	14	50	43	51	10	51	37	52	3	52	27	52	50	53	12	34
57	47	11	47	46	48	21	48	55	49	28	50	1	50	32	51	2	51	32	52	0	52	28	52	54	53	19	53	43	54	6	33
58	47	53	48	29	49	5	49	40	50	14	50	47	51	19	51	50	52	21	52	50	53	18	53	46	54	12	54	36	55	0	32
59	48	34	49	11	49	48	50	23	50	58	51	32	52	6	52	38	53	9	53	39	54	8	54	37	55	3	55	29	55	53	31
60	49	14	49	52	50	30	51	7	51	43	52	18	52	52	53	25	53	57	54	28	54	58	55	27	55	55	56	21	56	46	30
61	49	54	50	33	51	12	51	49	52	26	53	2	53	37	54	11	54	44	55	16	55	47	56	17	56	46	57	13	57	39	29
	29		28		27		26		25		24		23		22		21		20		19		18		17		16		15	

Compendium Calculi

PROBLEMA I.

Data basi cum uno angulo indagare latus dicto angulo oppositum.

IN Triangulo Rectangulo Sphærico A B C, angulum ad B rectum habente, data sit basis A B cum angulo C, vt habeatur latus A B dicto angulo oppositum. Ingredere Tabulam Primæ Analogiæ, quam Generalem vocamus, lateraliter cum basi & angulo datis, sumendo vnum ex his numeris frontaliter, & alium à latere sinistro: nam in areali vtriusque concursu obtinebis quæsitum latus.



EXEMPLVM.

Esto basis A C gr.64.3′o. & angulus C gr.32.2′o. facto cum his numeris laterali in primam tabulam ingressu, colligo in area pro latere quæsito A B, gr.28.5′2. ferè.

PROBLEMA II.

Datis duobus lateribus circa rectum basim inuenire.

DENTVR nunc Trianguli Rectanguli A B C duo latera A B. B C. circa angulum B rectum; & quæratur basis A C, quod est latus angulo recto oppositum.

Quo situm hoc per Generalem seu primam tabulam laterali introitu indagatur, comperiendo videlicet vnum ex lateribus datis in calce dictæ tabulæ, & aliud à latere dextro: & in area quidem elicietur complementum quæsitæ basis; vnde & ipsa statim per subductionem dicti inuenti numeri à gradibus 90. innotescet.

EXEMPLVM.

Data sint duo latera circa rectum, vt posse A B gr. 36. 3′o. B C gr.52. His itaque adinuentis in calce & à latere dextro excipio ex area gr.29.4′o. ferè, quibus sublatis à gradibus 90. prodit quæsita basis gr.60.2′o.

PROBLEMA III.

Dato latere cum angulo adiacente inquirere angulum illi lateri oppositum.

IN Triangulo Rectangulo Sphærico A B C, offeratur notum latus B C cum angulo C ei adiacente: vt ex his cognosci possit angulus A, qui dicto lateri opponitur, confugiemus ad eandem Primam Tabulam, & per lateralem introitum, quæsitum ipsum obtinebimus, ex area in hunc modum. Sumatur in dicta tabula frontaliter angulus datus, & à dextro latere latus datum, siue etiam sumatur à sinistro latere angulus datus, & in calce latus datum, nam vtroque modo respondet in area quæsiti anguli complementum, quo de quadrante sublato prodit ipse angulus.

EXEMPLVM.

Oblatum Triangulum habeat angulum C gr.42. cum latere B C adiacente gr.56.3′o. Sumo igi-

tur

Plate 27. C. A. Magini, Tabulae primi mobilis, with trigonometry, 1604.

Operationis summa. Sumantur tam Secantes secundæ, quàm Tangentes secundæ vtriusque anguli dati; lateris verò dati capiatur Sinus secundus. Multiplicentur primò inuicem Secantes secundæ amborum angulorum, & per totum Sinum fiat diuisio, & producetur Inuentum primum. Ex multiplicatione autem Tangentium secundarum vtriusque anguli, & per Sinum totum diuisione resultat Inuentum secundum; cum quo, & cum Sinu secundo lateris dati formatur Inuentum tertium iuxta varietatem casuum superioris Tabellæ. Hoc autem Inuentum tertium, vbi fuerit per integrum Sinum multiplicatum; &, quod hinc producitur, per totum Sinum diuisum; producetur Sinus secundus anguli quæsiti.

Exemplum primum congruens primo superiorum modorum.

		G	M	S		
Anguli dati	A	46	10	0	1 3 8 6 2 8	Secantes secundæ.
	B	60	0	0	1 1 5 4 7 0	
					1 6 0 0 7 4	Inuentum primum pro diuisore.
Anguli ijdem dati	A	46	10	0	9 6 0 0 8	Tangentes secundæ.
	B	60	0	0	5 7 7 3 5	
					5 5 4 3 0	Inuentum secundum.
Latus datum A B		8	0	0	9 9 0 2 7	Sinus eius secundus.
					4 3 5 9 7	Inuentum tertium multiplicandum in totum.
Angulus C quæsitus		74	11	49	2 7 2 3 5	Sinus secundus proueniens.

Exemplum secundum congruens secundo superiorum modorum.

Anguli dati	A	125	10	0	1 2 2 3 2 7	Secantes secundæ.
	B	25	0	0	2 3 6 6 2 0	
					2 8 9 4 5 0	Inuentum primum pro diuisore.
Anguli ijdem dati	A	125	10	0	7 0 4 5 5	Tangentes secundæ.
	B	25	0	0	2 1 4 4 5 1	
					1 5 1 0 9 1	Inuentum secundum.
Latus datum A B		52	30	0	6 0 8 7 6	Sinus eius secundus addendus.
					2 1 1 9 6 7	Inuentum tertium multiplicandum in totum.
Angulus C quæsitus		42	55	9	1 7 3 2 3 1	Sinus secundus proueniens.

Exemplum tertium congruens cum tertio superiorum modorum.

Anguli dati	A	60	0	0	1 3 8 6 2 8	Secantes secundæ.
	B	113	50	0	1 1 5 4 7 0	
					1 6 0 0 7 3	Inuentum primum proueniens.
Anguli ijdem dati	A	60	0	0	9 6 0 0 8	Tangentes secundæ.
	B	113	50	0	5 7 7 3 5	
					5 5 4 3 0	Inuentum secundum subtrahendum. Sinus lateris A B
Latus datum A B		172	0	0	9 9 0 2 7	Sinus eius secundus.
					4 3 5 9 7	Inuentum tertium multiplicandum in totum.
Angulus C quæsitus		105	48	16	2 7 2 3 6	Sinus excessus supra Quadrantem.

Quartus modus noster supradicto Vietæ conformis.

SVppletur à nobis prædicens Vietæ modus; vt nulla in eo interueniat tædiosa diuisio, tali ratione iuxta Theor. 41. Primi libri.

Enimuerò duo Rectangula figillatim applicabuntur ad Sinum totum. Vnum, quod fit sub Tangentibus secundis datorum angulorum; alterum sub Sinibus eorundemmet angulorum. Et erit, vt Sinus totus ad exeuntem è secunda applicatione latitudinem; ita aggregatum, vel differentia latitudinis ex prima applicatione oriundæ, & Sinus secundi lateris dati ad Sinum secundum anguli quæsiti.

Hoc est; fiat, vt Sinus totus ad Sinum vnius anguli dati; ita Sinus alterius anguli dati ad Inuentum primum; &, vt in superiore, per secundas Tangentes eorundemmet angulorum colligatur secundum Inuentum; quin etiam & tertium Inuentum similiter, vt iam expressimus, cognoscatur

Demum

Analogia huius Theorematis.

1 Quadratum Sinus totius.
2 Rectangulum sub Sinibus duorum laterum.
3 Sinus versus anguli à dictis lateribus comprehensi.
4 Differentia inter Sinuum versum tertij lateris dicto angulo oppositi, & Sinum versum differentiæ reliquorum laterum.

THEOREMA XXXVIII.

In Triangulo Obliquangulo, vt est Sinus totus ad quartam lineam proportionalem, seu ad quartum numerum proportionalem respectu totius Sinus, & Sinuum duorum laterum, ita est Sinus versus anguli à dictis lateribus comprehensi ad differentiam inter Sinum versum tertij lateris dicto angulo oppositi, & Sinum versum differentiæ reliquorum laterum.

THeorema hoc ex combinatione duorum præcedentium statim colligitur. Nam cùm ex Theremate trigesimo sexto Quadratum totius Sinus ad Rectangulum sub duobus Sinibus duorum laterum comprehensum eandem habeat proportionem, quam Sinus totus ad quartam proportionalem respectu totius Sinus, & duorum illorum Sinuum; cumq; Quadratum totius Sinus sit ad Rectangulum sub Sinibus duorum laterum cõprehensum; vt est Sinus versus anguli ab ijdem lateribus comprehensi ad differentiam duorum Sinuum verforum tertij scilicet lateris, & differentiæ reliquorum laterum ex præcedenti Theoremate. Igitur sequetur, quod propositum fuit.

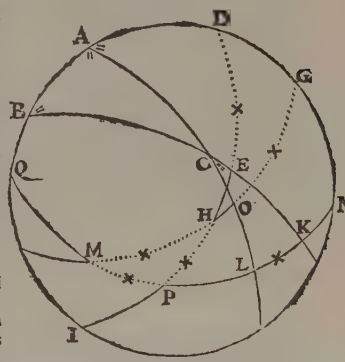
Analogia huius Theorematis.

1 Sinus totus.
2 Quartus terminus proportionalis respectu totius Sinus, & Sinuum duorum laterum.
3 Sinus versus anguli à dictis lateribus comprehensi.
4 Differentia inter Sinum versum tertij lateris dicto angulo oppositi, & Sinum versum differentiæ reliquorum laterum.

THEOREMA XXXIX.

In Triangulo Obliquangulo, vt est Quadratum totius Sinus ad Rectangulum sub Sinibus rectis duorum angulorum æqualium. Vel vt est Sinus totus ad quartum terminum proportionalem respectu totius Sinus, et Sinuum rectorum duorum angulorum æqualium, ita est Sinus versus lateris vtriq; angulo adiacentis ad differentiam duorum Sinuum versorum, quorum vnus est tertij anguli, & alter est differentiæ dictorum angulorum vnius, & alterius ad semicirculum complementi.

SIt propositum Triangulum Obliquangulum A B C, cuius anguli B, & A sint inæquales. Dico Quadratũ totius Sinus esse ad Rectangulum Sinuum rectorũ duorum angulorum A, & B. Vel Sinum totũ esse ad quartum terminum proportionalem respectu totius Sinus, & Sinuum duorum angulorum B, & A, vt est Sinus versus lateris A B ad differentiam duorum Sinuum verforum, quorum vnus est anguli C, alter verò est differentiæ inter vnum angulum, puta B, & complementum alterius anguli A ad semicirculum. Fiat ipsius Trianguli A B C commutatio, seu inuersio in Triangulum H M P lateribus, & angulis illi reciprocum, iuxta ea, quæ attulimus supra numero decimosexto. Enimuerò pro angulo B propositi Trianguli dabitur latus P H inuersi Trianguli; & pro complemento anguli A ad semicirculum dabitur latus M H, & tertio angulo C ipsius propositi Trianguli respondebit latus M P.



Præterea lateri A dicti Trianguli propositi æqualis est angulus H inuersi Trianguli.

G

Quo-

Plate 28. G. A. Magini. Pages from his Primum Mobile, Bologna, 1609. With Trigonometry.

BD, ACD, nisi uterq; rectus sit, vt demonstrabimus. Hoc autem ideo euenit, quod non uterque angulus ABD, ACD, maior est vel minor recto, sed vel uterque rectus, vel unus maior recto, & alter minor: quod ita ostendemus. Sit primum angulus ABD, [h] *rectus. Dico & C, rectum esse. Recto enim existente angulo ABD, erit & ABC, rectus;* [k] *quod ambo anguli ad B, æquales sint duobus rectis:* [l] *sed hic æqualis est angulo C, ob æqualitatem laterum AB, AC. Igitur & C, rectus erit.*

h 5. huius. i 8. huius. k 5. huius. l 8. huius.

SIT deinde angulus ABD, maior recto. Dico C, minorem esse recto. Cum enim ABD, sit recto maior, erit ABC, minor recto, [k] *cum ambo duobus rectis sint æquales. Igitur & angulus C,* [l] *qui æqualis est angulo ABC, recto minor erit.*

SIT tandem angulus ABD, minor recto. Dico C, esse recto maiorem. Cum enim ABD, sit minor recto, erit ABC, hoc est, sibi æqualis C, maior recto.

HINC manifestum est, propositionem 8. Nicolai Copernici de sphæricis triangulis, lib. 1. Reuolutionum falsam esse, quo ad eam partem, in qua dicit. Si bina triangula duo latera duobus lateribus æqualia habuerint, alterum alteri, & angulum angulo æqualem, qui ad basim fuerit; basim quoque basi, ac reliquos angulos reliquis angulis habebunt æquales. *Hoc enim verum non est, nisi ponatur uterque reliquorum angulorum ad basim vel maior recto, vel minor. In triangulis enim propositis ADB, ADC, sunt duo latera AD, AB, duobus lateribus AD, AC, æqualia, angulus; D, communis est super bases DB, DC; & tamen bases non sunt æquales, ob causam dictam.*

Error Nicolai Copernici.

INDE errat idem Nicolaus in eodem libro propos. 11. vbi ait. Omne triangulum, cuius duo latera fuerint data cum aliquo angulo, datorum efficitur angulorum, & laterum. *Nam etiamsi latera AD, AB, nota sint cum angulo D, non tamen inde in notitiam alterius lateris, & aliorum angulorum perueniemus, cum reliquum latus possit esse vel DB, vel DC, &c. Necesse est ergo aliud quippiam præterea conflare, antequam reliquum latus, cum reliquis angulis notum efficiatur, vt in Scholio propos. 67. perspicuum faciemus.*

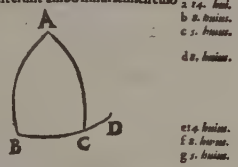
Alius error Nicolai Copernici.

THEOR. 23. PROP. 25.

IN omni triangulo sphærico Isoscele, si duo latera æqualia sint quadrantes, erunt duo anguli æquales super basim recti: si vero vtrumque quadrante minus sit, acuti: si denique maius quadrante, obtusi. Et si duo anguli æquales ad basim sint recti, erunt duo latera æqualia quadrante: si vero acuti, vtrumque quadrante minus erit: si denique obtusi, vtrumq; quadrante maius.

IN triangulo sphærico Isoscele ABC, sint primum duo arcus æquales AB, AC, quadrantes. Dico æquales angulos B, C, ad basim esse rectos. Cum enim vterq; arcus AB, AC, quadrans sit erunt ambo simul semicirculo æquales. Quare producto arcu BC, ad D, [a] angulus ACD, æqualis erit angulo B: [b] sed angulus B, angulo ACB, æqualis est. Igitur & angulus ACD, angulo ACB, æqualis erit; atq; adeo, [c] cum duo anguli ad C, duob. rectis æquales sint, erit vterq; angulus ad C, rectus. Quare & angulus B, [d] qui recto ACB, æqualis est, rectus erit. Quod est propositū.

a 14. huius. b 8. huius. c 5. huius. d 8. huius.



SIT deinde vterque arcuum AB, AC, æqualium quadrante minor. Dico angulos B, C, æquales esse acutos. Cum enim vterque arcus AB, AC, quadrante minor sit, erunt ambo simul semicirculo minores. [e] Quare angulus ACD, maior erit angulo B, hoc est, angulo ACB, [f] cum anguli B, & ACB, æquales sint. [g] Cum ergo duo anguli ad C, æquales sint duobus rectis, erit angulus ACB, recto minor; atque adeo angulus B, [h] qui ei æqualis est, recto quoque minor erit. Sunt ergo duo anguli B, & ACB, acuti. Quod est propositum.

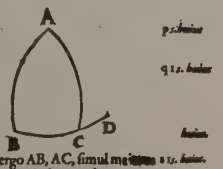
e 14. huius. f 8. huius. g 5. huius. h 8. huius.

SIT postremo vterque arcuum AB, AC, quadrante maior. Dico angulos æquales, B, C, esse obtusos. Cum enim vterque arcus AB, AC, maior sit quadrante, erunt ambo maiores semicirculo. [i] Quare angulus ACD, minor erit angulo B, hoc est, angulo ACB, [k] qui angulo B, æqualis est. [l] Cum ergo duo anguli ad C, duobus rectis sint æquales, erit angulus ACB, recto maior, hoc est, obtusus; atque idcirco & angulus B, [m] qui ei æqualis est, obtusus erit. Quod est propositum.

i 14. huius. k 8. huius. l 5. huius. m 8. huius.

SED iam vterque angulorum æqualium B, C, sit rectus. Dico vtrumq; arcum AB, AC, quadrantem esse. Cum enim ACB, rectus sit, [n] & duo anguli ad C, æquales duobus rectis, erit quoque ACD, rectus, ac proinde recto B, æqualis. [o] Sunt ergo duo arcus AB, AC, simul semicirculo æquales, ac propterea cum ipsi æquales ponantur, vterque quadrans erit. Quod est propositum.

n 5. huius. o 15. huius.



DEINDE vterq; angulorum B, C, sit acutus. Dico vtrumq; arcum AB, AC, quadrante minorem esse. [p] Cum enim duo anguli ad C, æquales duobus rectis sint, & ACB, ponatur recto minor; erit ACD, recto maior; ac propterea maior, quam B, qui recto etiam minor ponitur. [q] Sunt ergo arcus AB, AC, simul semicirculo minores; atq; idcirco, cum ipsi sint æquales, vterque quadrante minor erit. Quod est propositum.

p 5. huius. q 15. huius.

POSTREMO sit vterque angulorum B, C, obtusus. Dico vtrumque arcum AB, AC, maiorem esse quadrante. [r] Cum enim duo anguli ad C, sint æquales duobus rectis, & ACB, ponatur maior recto, erit ACD, recto minor, atque idcirco minor angulo B, qui recto quoque maior ponitur. [s] Arcus ergo AB, AC, simul me[illegible] sunt semicirculo; atque adeo, cum ipsi æquales sint, erit vterque quadrante maior. Quod est propositum. In omni ergo triangulo sphærico Isoscele, &c. Quod demonstrandum erat.

r 5. huius. s 15. huius.

COROLLARIVM.

EX his sequitur, omne triangulum sphæricum æquilaterum, seu æquiangulum, si singula latera sint quadrantes, habere singulos angulos rectos: si vero quadrante minora, acutos. Si deniq; quadrante maiora, obtusos. Et omne triangulum sphæricum æquiangulum, seu æquilaterum, si singuli anguli sint recti, habere singula latera quadrantes: Si vero acuti, quadrante minora: si deniq; obtusi, quadrante maiora.

Q SCHO-

SCHOLIVM.

Data duo huius anguli trianguli rectilinei dantur etiam reliqui. Item si in triangulo rectilineo dentur vnus angulus acutus, datum quoq; erit acutus reliquus. Latera trianguli rectilinei sunt sinibus angulorum oppositorum proportionalia. 25. primi.

IN materia porro triangulorum rectilineorum, cum dantur duo anguli noti, tertius illico notus quoque erit, cum sit complementum duorum reliquorum. Item cū in triangulo rect. angulo datur vnus acutus angulus, notus etiā erit reliquus acutus, quod sit complementum unius recti. Itaq, detractus duobus angulis notis simul ex grad. 180. reliquus erit tertius notus. Item in triangulo rectangulo, si detrahatur acutus notus ex grad. 90. rem. nobis datur acutus notus. Quod semel monuisse satu fit.

THEOR. 1. PROPOS. 1.

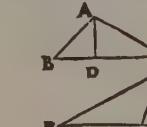
IN omni triangulo rectilineo latera quæuis duo eandem proportionem habent, quam sinus angulorum illis oppositorum.

SIT primum triangulum rectangulum ABC, cuius angulus rectus B. Dico esse AB, ad AC, vt est sinus anguli C, ad siuum anguli B. Item AB, ad BC, vt est sinus anguli C, ad sinum anguli A, &c. Quoniam enim, vt in definitionibus sinuum ostendimus, si AC, ponatur sinus totus, latus AB, erit sinus anguli C, & BC, sinus anguli A: liquido constat, ita esse latus AB, ad latus AC, vt est AB, sinus anguli C, ad AC, sinum totum anguli recti B: Vel ita esse latus AC, ad latus AB, vt est AC, sinus totus recti anguli B, ad AB, sinum anguli C; cum ipsa latera sint sinus angulorum oppositorum, ac proinde vtrobique sit identitatis proportio. Eadem ratione erit, vt latus AC, ad latus BC, ita AC, sinus totus anguli recti B, ad BC, sinum anguli A: Vel vt latus BC, ad latus AC, ita BC, sinus anguli A, ad AC, sinum totum recti anguli B. Item vt latus AB, ad latus BC, ita AB, sinus anguli C, ad BC, sinum anguli A: Vel vt latus BC, ad latus AB, ita BC, sinus anguli A, ad AB, sinum anguli C.

SIT deinde triangulum ABC, non rectangulum. Dico rursus esse latus AB, ad latus AC, vt est sinus anguli C, ad sinum anguli B, &c. Aut enim latera assumpta AB, AC, æqualia sunt, aut inæqualia. Si æqualia, [a] erunt quoque anguli C, B, æquales; ac proinde, vt in definitionibus sinuum docuimus, eorum sinus æquales. Quare erit, vt latus AB, ad latus AC, ita sinus anguli C, ad sinum anguli B, vel latus AC, ad latus AB, ita sinus anguli B, ad sinum anguli C; cum semper sit proportio æqualitatis. Si vero latera AB, BC, sunt inæqualia, sic AC, maius, ex quo abscindatur recta CE, minori lateri AB, æqualis, & ex A, E, ad tertium latus BC, perpendiculares demittantur AD, EF, quarum vtraque cadet intra triangulum, quando angulus B, maiori lateri AC, oppositus acutus est. Erit enim & tunc angulus quoque C, acutus, [b] cum minor sit, quam B. [c] Quare perpendicularis AD, intra triangulum cadet, ac proinde & perpendicularis EF. Quando vero angulus B, obtusus est, [d] cadet quidem AD, semper extra triangulum, at EF, cadere potest vel extra etiam, vel in punctum B, vel intra triangulum. Quomodocunque autem cadant dictæ perpendiculares, semper eadem erit demonstratio. [e] Nam cum AD, EF, sint parallelæ, [f] erunt triangula CEF, CAD, similia. [g] Quamobrem erit, vt CE, ad EF, ita CA, ad AD. Cum ergo ex iis, quæ in definitionibus sinuum tradidimus, posito sinu toto CE, recta EF, sit sinus anguli C; posito item sinu toto AB, sit sinus anguli ABD; sintq; sinus toti CE, AB, respectu quorū illi sunt sinus, æquales; liquet esse, vt CE, hoc est, latus rectæ AD, AB, ad EF, sinum anguli C, ita latus CA, ad AD, sinum anguli ABD: Et permutando, vt latus AB, ad latus AC, ita EF, sinum anguli C, ad AD, sinum anguli ABD, hoc est, in posteriori triangulo, ad sinum anguli ABC, cum duo anguli ad B, æquales sint duobus rectis, & proinde eundem sinum habeant, vt in definitionibus sinuum docuimus. Ex quo constat, ita esse minus latus AB, ad maius AC, vt est EF, sinus anguli C, minori lateri opposīti ad AD, sinum anguli ABC, maiori lateri oppositi: Et conuertendo, ita esse maius latus AC, ad minus AB, vt est AD, sinus anguli ABC, maiori lateri oppositi ad EF, sinum anguli C, minori lateri oppositi. Non aliter ostendemus esse, vt latus AB, ad latus BC, ita sinum anguli C, ad sinum anguli A: Vel vt latus BC, ad latus AB, ita sinum anguli A, ad sinum anguli C, &c. dummodo ex puncto, vbi conueniunt latera assumpta inæqualia, (si forte æqualia non sunt) ducas ad latus oppositum lineam perpendicularem, & minori lateri ex maiore rectam æqualem abscindas, initio facto ab altero puncto extremo maioris lateris, vbi cum tertio latere coniungitur, vt a nobis factum est &c. Hic etiam est, vt CE, hoc est, AB, ad CA, ita EF, sinus anguli C, oppositi priori lateri AB, ad AD, sinum anguli B, oppositi priori lateri AC. Dico enim angulī ad B, eundem sinum habent.

[a] 14. huius. [b] 12. pri. [c] schol. 13. secundi. [d] schol. 12. secundi. [e] 12. pri. Coroll. 4. sexti. [f] 4. sexti. [g] 4. sexti.

ALITER. Sit rursus triangulum non rectangulum ABC: de rectangulo enim in principio huius demonstrationis iam est demonstratum. Dico esse, vt latus AB, ad latus AC, ita sinum anguli C, ad sinum anguli B: Vel vt latus AC, ad latus AB, ita sinum anguli B, ad sinum anguli C, &c. Ducta enim ex A, vbi duo latera assumpta cocunt, ad tertium latus BC, perpendiculari AD, quæ vel intra triangulum cadet, vel extra, prout anguli B, & C, acuti fuerint, vel alter eorum obtusus: erit in triangulo rectangulo ABD, vt latus AB, ad latus AD, ita sinus anguli recti D, ad sinum anguli B, vt supra est demonstratum: Item in triangulo rectangulo ADC, vt latus AD, ad latus AC, ita sinus anguli C, ad sinum anguli recti D. Ex æqualitate ergo, & permutata



latus AB, ad latus AC, sinus anguli C
latus AD, ad sinum D.
latus AC, ad sinum B.

propor-

10. ADRIANUS ROMANUS

Two trigonometric works[130] are attributed to Romanus.[131] The first treatise, Ideae mathematicae pars prima, published at Antwerp in 1593, is dedicated to Christopher Clavius.[132] It contains the solution of the famous forty-five degree equation which Romanus proposed to Vieta; the work, in general, is concerned with the division of a circle and the calculation of π.[133] The second treatise follows a plan more in keeping with the trigonometric texts of the period. It is entitled Canon triangulorum and was published at Mainz in 1607 and 1609.[134] In this work Romanus reduces the twenty-eight formulas relative to right-angled triangles to six, employs the sine, cosine, and cotangent theorems for spherical triangles, and also handles the polar formulas.[135] In his terminology he imitates Vieta, using the expression "prosinus" and "transsinuosa" for the tangent and secant respectively. The special notation that he introduces for the trigonometric functions will be found in the table at the end of Chapter V. His tables for sines, tangents, and secants together with their co-functions are calculated from 0° to 90° for the radius 1000000000.[136]

11. THOMAS BLUNDEVILLE

One looks in vain to England during this period for a trigonometric text with the characteristics of those of the continental mathematicians. The absence of a trigonometric treatise is not indicative of lack of interest in trigonometry, as a quotation from the preface to the Exercises[137] of Thomas Blundeville[138] can verify.

> I haue thought it good to adde vnto mine Arithmeticke, as an appendix depending thereon, the vse of the Tables of the three right lines belonging to a circle, which lines are called Sines, lines Tangent, and lines Secant, whereby many profitable and necessarie conclusions, as well of Astronomie, as of Geometrie are to bee wrought onely by the helpe of Arithmeticke, which tables are set down by Clauius the Iesuite, a most excellent Mathematician, in his Booke of demonstrations made vpon the Spheriques of Theodosius, more truely printed than those of Monte Regio, which Booke whiles I read at mine owne house, together with a Louing Friend of mine, I took such delight therein, as I mind (God willing) if God giue me life, to translate all those propositions which Clauius himselfe hath set down of his owne, touching the quantity of Angles, and of their sides, as well in right line Triangles, as in the Spherical Triangles, of which matter, as Monte Regio wrote diffusedly, and at large, so Copernicus wrote of the same briefly but therewith somewhat obscurely, as Clauius saith.

Besides giving the tables of sines, tangents, and secants, Blundeville defines and illustrates these functions in a manner similar to that of Blagrave.[139] He also proposes problems indicating whether they are to be solved by using the tables of Regiomontanus or those of Clavius.

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130. These works of Romanus are not available at U.M.
131. Adrianus Romanus was born at Louvain in 1561 and became professor of mathematics there in 1586. In 1609 he visited Vieta at Tours; a life long friendship existed between these two mathematicians. Romanus died in 1615 (A. von Braunmühl, op. cit., p. 173).
132. Idem.
133. Ibid., p. 174.
134. In 1606 Romanus published Speculum astronomicum which is essentially an astronomical text.
135. A. von Braunmühl, op. cit., p. 229.
136. Ibid., p. 230.
137. M. Blundevil: His Exercises Contayning Six Treatises... London, 1594. Other editions are: 1597, 1606, 1613, 1622, 1631; the last two editions are at U.M. In all editions except the first there are eight exercises instead of six.
138. Thomas Blundeville (fl. 1561) is supposed to have been educated at Cambridge, though it is not certain at what time or in what house he resided (Dictionary of National Biography, vol. V, p. 271).
139. See pp. 90-91.

12. BARTHOLOMÄUS PITISCUS

The word "trigonometry" originated with Bartholomäus Pitiscus[140] who used it in his Trigonometriae sive De dimensione triangulorum libri quinque, published for the first time in 1595 at Heidelberg with Scultetus'[141] Sphaericorum libri tres. The revised editions of the Trigonometriae issued independently of Scultetus' work are the Augsburg editions of 1600[142] and 1608 and the Frankfort edition of 1612. An English translation of the Trigonometriae was made by Handson and edited in 1614; the other editions of this translation were published in 1630 and 1631.

The 1600 edition comprises five books on plane and spherical trigonometry; tables for sines, tangents, and secants, and ten books[143] treating of problems in geodesy, measuring of heights, geography, gnomonetry and astronomy complete the text.

The first book considers definitions and theorems from plane and spherical geometry, such as: "Trigonometria est doctrina de dimensione Triangulorum," and

> A triangle is a figure with three sides including three angles such as figures ABC and DEF.[144]

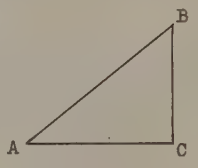

Fig. 126(a)

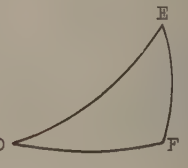

Fig. 126(b)

The second books is concerned with the things that must be known in order to solve triangles by means of the tables of sines, tangents, and secants. Pitiscus states that the dimensions of a triangle can be determined if three parts are already known by applying the "golden rule" of arithmetic - if four things are proportional among themselves, then given three, the fourth can be determined. This book includes the definitions of the trigonometric functions, a method for constructing the trigonometric tables, and the fundamental trigonometric identities.

Book III is devoted to plane trigonometry. What other authors designated propositions or theorems, Pitiscus calls axioms. A few of his axioms are: Axiom 1: "The hypotenuse is to the perpendicular as the radius is to the sine and conversely." This axiom is proved by noting the corresponding parts of similar triangles.

Axiom 4 is the Sine Law; the proof and figure are the same as those of Regiomontanus.

Axiom 5: "The sum of two sides of a triangle is to their difference as the tangent of half the sum of the two angles opposite the two sides is to the tangent of the difference of the two angles."

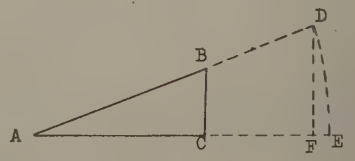

Fig. 127

Pitiscus uses this axiom to solve a triangle when two sides and the included angle are given. Since his proof differs from the one given by Finck in that it applies to a specific triangle, it will be produced here. The device of marking the given parts

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140. Bartholomaus Pitiscus was born in Silesia in 1561. He was court chaplain at Breslau and teacher of Prince Frederic IV at Heidelberg. He died in 1631 (A. von Braunmühl, op. cit., p. 221).

141. The astronomer Scultetus or Schulz was in his youth a friend of Tycho Brahe and studied mathematics at Leipzig and Wittenberg (A. von Braunmühl, op. cit., p. 196).

142. This edition is at U.M.

143. The 1612 edition gives eleven, adding one on architecture (A. von Braunmühl, op. cit., p. 225).

144. Magini followed Pitiscus in stating this definition simultaneously for plane and spherical triangles. Magini's figures, however, are not right triangles.

THEODOSII TRIPOLITÆ SPHÆRICORVM LIBRI III.

à

CHRISTOPHORO CLAVIO BAMBERGENSI SOCIETATIS IESV

PERSPICVIS DEMONSTRATIONIBVS, & ſcholiis illuſtrati.

Et ab eodem ſecunda hac editione correcti, & aucti.

MOGVNTIÆ,

Sumptibus ANTONII HIERAT, excudebat REINHARDVS ELTZ.

Cum Gratia & Priuilegio Sacræ Cæſar. Maieſt.

ANNO M. DC. XI.

Plate 30. Theodosius of Tripoli. Sphaericorum Libri III, edited by Clavius, 1611.

THESAVRVS MATHEMATICVS

sive

CANON SINUUM AD RADIUM

1. 00000. 00000. 00000.

ET AD DENA QVÆQVE SCRVPVLA
secunda Quadrantis:

UNA CUM SINIBUS PRIMI ET POSTREMI GRADUS, AD EVNDEM RADIVM, ET AD SINGVLA

scrupula secunda Quadrantis:

ADIVNCTIS VBIQVE DIFFERENTIIS PRIMIS ET SECUNDIS, atq; vbi res tulit, etiam tertijs.

IAM OLIM QVIDEM INCREDIBILI LABORE & sumptu à GEORGIO IOACHIMO RHETICO supputatus:

AT NVNC PRIMVM IN LVCEM EDITVS, & cum viris doctis communicatus

A

BARTHOLOMÆO PITISCO
GRUNBERGENSI SILESIO.

CVIVS ETIAM ACCESSERVNT:

I. Principia Sinuum, ad radium, 1.00000.00000.00000.00000.00000.quàm accuratissimè supputata.

II. Sinus decimorum, tricesimorum & quinquagesimorum quorumq; scrupulorum secundorum per prima & postrema 35. scrupula prima, ad radium, 1.00000.00000.00000.00000.00.

FRANCOFURTI
Excudebat Nicolaus Hoffmannus, sumptibus
IONÆ ROSÆ ANNO

cIↃ. IↃ. XIII.

Plate 31. Georg Joachim Rheticus, Thesaurus Mathematicus, Frankfort, 1613. The sines to 15 places.

AD RADIVM

22	Sinus	Diff. I.	II.	III.	Sinus complementi	Diff. I.	II.	III.	
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30	38805.183390.83451	4468179317 3	912095	102	92163.75481.63023	1881220609 21	2160257	46	10
10	38809.65208.76624	4468088 0974	912197	108	92161.87337.95845	1881436717 8	2160212	45	50
20	38814.12017.57598	4467996 83670	912305	103	92159.99172.62455	1881653319	2160168	44	40
30	38818.58817.26268	4467905 6161	912408	106	92158.10985.62897	1881870558	2160123	45	30
40	38823.45607.82530	4467814 3768	912514	105	92156.22776.97216	1882086 5681	2160082	41	20
50	38827.52389.26278	4467723 1129	912619	105	92154.34546.65453	1882303 1763	2160033	49	10
31	38831.99161.57407	4467631 8405	912724	104	92152.46294.67657	1882521 97796	2165994	49	9
10	38836.45924.75812	4467540 5577	912828	107	92150.58021.03867	1882736 3790	2165945	49	50
20	38840.92678.81389	4467449 2642	912935	104	92148.69725.74133	1882952 9736	2165905	42	40
30	38845.39423.74031	4467357 9604	913039	104	92146.81408.78494	1883169 5639	2165857	46	30
40	38849.86159.53635	4467266 6460	913143	106	92144.93070.16999	1883386 1495	2165816	41	20
50	38854.32886.20094	4467175 3211	913249	106	92143.04709.89688	1883602 7311	2165769	47	10
32	38858.79603.73305	4467083 9856	913355	103	92141.16327.96608	1883819 3080	2165726	43	8
10	38863.26312.13162	4466992 6598	913458	106	92139.27924.37802	1884035 8806	2165681	45	50
20	38867.73011.39559	4466901 2834	913564	104	92137.39499.13315	1884252 4487	2165637	44	40
30	38872.19701.52394	4466810 9166	913668	106	92135.51052.23191	1884469 0124	2165592	45	30
40	38876.66382.51560	4466718 5392	913774	106	92133.62583.67475	1884685 5716	2165550	42	20
50	38881.13054.36951	4466627 1512	913880	103	92131.74093.46209	1884902 1266	2165503	47	10
33	38885.59717.08463	4466535 7529	913983	106	92129.85581.59440	1885118 6769	2165459	45	7
10	38890.06370.65992	4466444 3440	914089	105	92127.97048.07212	1885335 2218	2165417	42	50
20	38894.53015.09433	4466352 9246	914194	104	92126.08492.89568	1885551 7645	2165371	46	40
30	38898.99650.38679	4466261 4948	914298	107	92124.19916.06551	1885768 3016	2165326	45	5
40	38903.46276.53626	4466170 0543	914405	102	92122.31317.48209	1885984 8342	2165284	42	30
50	38907.92893.54170	4466078 6036	914507	108	92120.42697.44584	1886201 3626	2165236	48	10
34	38912.39501.40205	4465987 1421	914615	103	92118.54055.65721	1886417 8862	2165196	40	9
10	38916.86100.11626	4465895 6703	914718	107	92116.65392.21663	1886634 4058	2165148	48	50
20	38921.32689.68329	4465804 1878	914825	102	92114.76707.12457	1886851 9206	2165106	42	40
30	38925.79270.10208	4465713 6951	914927	107	92112.88000.38145	1887067 4312	2165060	46	30
40	38930.25841.37158	4465621 1917	915034	104	92110.99271.98773	1887283 9372	2165017	43	20
50	38934.72403.49076	4465529 6779	915138	108	92109.10521.94384	1887500 4389	2164973	44	10
35	38939.18956.45854	4465438 1533	915246	100	92107.21750.25022	1887716 9362	2164927	46	5
10	38943.65500.27388	4465346 6187	915346	108	92105.32956.90734	1887933 4289	2164883	44	50
20	38948.12034.93575	4465255 0733	915454	104	92103.44141.91562	1888149 9172	2164839	44	40
30	38952.58560.44307	4465163 4175	915558	106	92101.55305.87550	1888366 4011	2164794	45	30
40	38957.05076.78482	4465072 0511	915664	103	92199.66446.98745	1888582 8805	2164751	43	20
50	38961.51583.98993	4464980 7744	915767	108	92197.77567.05189	1888799 3556	2164705	46	10
36	38965.98082.02737	4464888 7869	915875	102	92195.88665.46928	1889015 8261	2164663	42	4
10	38970.44570.90607	4464797 1893	915977	108	92193.99742.24005	1889232 2924	2164615	48	50
20	38974.91050.62499	4464705 5808	916085	102	92092.10797.36465	1889448 7539	2164574	41	40
30	38979.37521.18307	4464613 9621	916187	106	92090.21830.84353	1889665 2113	2164526	48	30
40	38983.83981.57928	4464523 3328	916393	104	92088.32842.67713	1889881 6639	2164486	40	20
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37	38992.76877.88187	4464339 0327	916504	103	92084.54801.41025	1890314 5563	2164396	42	3
10	38997.23311.78614	4464247 3820	916607	107	92082.65748.31066	1890530 9959	2164350	46	50
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30	39006.16152.09540	4464064 0289	916817	106	92078.87577.18142	1890963 8615	2164261	45	30
40	39010.62558.49829	4463972 3366	916923	103	92076.98459.16266	1891180 2876	2164218	43	20
50	39015.08955.73196	4463880 6340	917026	108	92075.09319.48172	1891396 7094	2164171	47	10
38	39019.55343.79535	4463788 9206	917134	102	92073.20158.16907	1891613 1265	2164130	41	2
10	39024.01722.68741	4463697 1970	917236	107	92071.30975.21512	1891829 5395	2164082	48	50
20	39028.48092.40718	4463605 4627	917343	104	92069.41770.63035	1892046 9477	2164041	41	40
30	39032.94452.95338	4463513 7180	917447	105	92067.52544.38517	1892262 3518	2163993	48	30
40	39037.40804.32518	4463421 9628	917552	105	92065.63296.51006	1892478 7511	2163952	41	20
50	39041.87146.52146	4463330 1971	917657	105	92063.74026.99543	1892695 1463	2163904	48	10
39	39046.33479.54117	4463238 4209	917762	103	92061.84735.84176	1892911 5367	2163862	42	1
10	39050.29803.38326	4463146 6344	917865	119	92059.95423.04947	1893127 9228	2163817	45	50
20	39055.26118.04670	4463054 8370	917974	102	92058.06088.61901	1893344 3046	2163773	44	40
30	39059.72423.53040	4462963 0294	918076	106	92056.16732.55082	1893560 6819	2163726	47	30
40	39064.18719.83334	4462871 2118	918182	103	92054.27354.84537	1893777 0545	2163685	41	20
50	39068.65006.95446	4462779 3827	918285	107	92052.37955.50307	1893994 4230	2163638	47	10
	Sinus complementi	Diff. I.	II.	III.	Sinus	Diff. I.	II.	III.	0

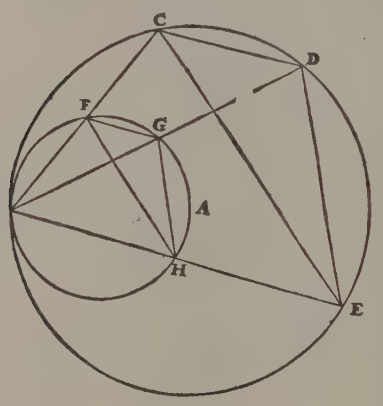
Plate 32. Georg Joachim Rheticus, Thesaurus Mathematicus, Frankfort, 1613. The sines to 15 places.

rectanguli dato, reliquum latus dari, hoc est, reliquam diametrum in figuræ, quatuor laterum circulo inscriptæ. Sed si vnum quatuor laterum desiderabitur, de rectangulo quod diametri figuræ includunt, auferatur rectangulum quod quæcunq; ex aduerso datæ figuræ latera includunt, & remanebit rectangulum, quod reliqua duæ figuræ latera incluserint. Quare altero rectanguli dato latere, dabitur & reliquum ignotum exquirendum latus.

LEMMA QVINTVM.

Dimidiatæ rectæ quatuor laterum figuræ in circulo inscriptæ, constituunt quatuor laterum figuram circulo circumscribendam.

Ad centrum A descriptus sit circulus BCD, & in eum fortuitò sit inscripta figura quatuor laterum BCDE, & in ea ducatur diameter BD. Deinde secantur bifariam rectæ BC in F, BD recta in G, & BE in H puncto, & iungantur GF, item GH. Dico quatuor laterum figuram BFGH, constare ex dimidiatis rectis quatuor laterum figuræ BCDE inscriptæ, & BFGH quatuor laterum figuram posse circumscribi circulo. Per constructionem in proportione sectæ sunt BC & BD in punctis F, G. Ergo FG, CD rectæ sunt parallelæ. Angulus igitur BFG, æqualis est angulo BCD, & vtriqué Triquetro communis est, CBD. Angulus igitur tertius BGF tertio BDE æqualis erit. Duo igitur Triquetra BCD, BFG angulos æquales habent: proinde sicut se habet BC latus ad FF latus, ita se habet CD latus ad FG latus. Sed BC est duplum rectæ BF per constructionem. Ergo & CD latus duplum est rectæ FG. Similiter demonstrabimus non tantum angulos BHG, BED, esse æquales, sed & angulos BGH, BDE: & latus DE duplum rectæ, GF.



Constat igitur figura quatuor laterum BFGH ex dimidiatis rectis laterum figuræ BCDE circulo inscriptæ. Demonstratum est etiam BFG, BCD angulos esse æquales, itemque BHG, & BED. Sed anguli BCD, BED sunt duobus rectis æquales. Ergo & anguli BFG, BHG ex aduerso duobus rectis erunt æquales, cumque duo anguli BGF, BGH sint æquales duobus angulis BCD, BED. Totus igitur angulus FGH, toti CDE angulo est æqualis. Assumatur ad vtrunque communis angulus CBE. Duo igitur anguli CDE, CBE æquales sunt duobus FGH, FBH angulis. Sed CDE, CBE anguli duobus rectis sunt æqules. Duo igitur FGH, FBH anguli, duobus rectis sunt æquales. Demonstratum verò est & angulos BFG, BHG duobus rectis esse æquales. In figura igitur quatuor laterum BFGH, anguli ex aduerso duobus rectis sunt æquales. Quare si figuræ quatuor laterum BFGH, circulus circumscribatur, ambitus eius figuræ angulos omnes attinget. Sed demonstratum est, singula latera huius figuræ esse semisses singulorum laterum illius, sic vt singula singulis respondeant. Dimidiatæ igitur rectæ quatuor laterum figuram circulo circumscribendam constituunt

ACQVISITVM RRIMVM.

Ex hoc Lemmate manifestum est, Ptolemaicum præcedens, cum suo acquisito eundem vsum habere, siue cum subtensis integris, siue cum earum semissibus ratiocinium sit instituendum. Siquidem similiter constabit, reliquas figuræ diametros in dupla ratione esse.

Plate 33. Georg Joachim Rheticus, Thesaurus Mathematicus, Frankfort, 1613. The sines to 15 places.

of a triangle is also shown. Pitiscus uses a different system for marking spherical triangles.[145]

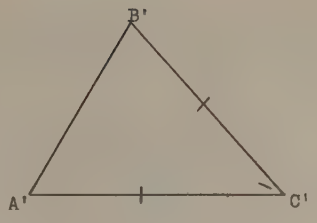


Fig. 128

A'B'C'[146] is the given triangle; the strokes indicate that the sides A'C' and B'C' and the angle C' are known.

> In a quadrant ABC construct angles DAE and EAC equal to angles A'B'C' and B'A'C' respectively. DAC will be the sum of the two angles; DAF or FAC will be half the sum. The difference between angle DAE and angle DAF is FAE. The chord of the sum of the two angles is DC. DG is the sine of DAE, CH is the sine of EAC. The tangent of half the sum is FM or FK; the tangent of the difference is FE. Draw AL so that angle KAL equals angle MAE.

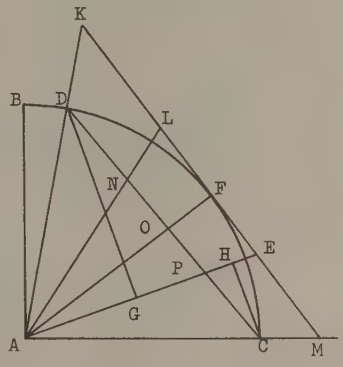


Fig. 129

Triangle GDP and CPH are equiangular, so it follows that PD/GD = PC/CH, i.e., PD/sin DAE = PC/sin EAC. Considering Fig. 128 and applying the sine law, it is true that A'C'/sin A'B'C' = B'C'/sin B'A'C'; because angle A'B'C' equals angle DAE and angle B'A'C' equals angle EAC, PD/PC = A'C'/B'C'. Therefore the ratio of the sum and difference of PD and PC is equal to the ratio of the sum and difference of A'C' and B'C'.

DP + PC = DC and DP - PC = NP. Triangle AKM and ADC are similar because DC and KM are parallel, so DC/NP = KM/LE whence OC/OP = FM/FE, i.e., half the sum of the two sides is to the half their difference as the tangent of half the sum of the two angles opposite the sides is to half the tangent of their difference.

Axiom 6 is the statement and proof of the Cosine Theorem for a plane triangle. The spherical triangle is considered in Book IV. The first axiom states that the sines of hypotenuses and perpendiculars of right spherical triangles of the same sphere are proportional; the second that the sines of the bases and tangents of the perpendiculars are proportional; the third is the Sine Law; and the fourth is the Cosine Theorem for which Pitiscus was the first to give a real proof for the theorem relative to angles; Lansberg[147] merely stated it.

By means of these four axioms Pitiscus solves right and oblique spherical triangles. The device he uses in marking the given and required parts in this book are illustrated by the following problem:

> If an oblique triangle ABC having the angles A and B and the side AB given, investigate the side AC or BC and angle C by means of Axiom 4 [cosine theorem] and the side AC or BC by the third axiom [sine law].

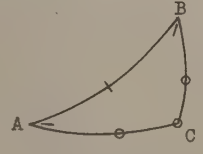


Fig. 130

Pitiscus does not consider the polar triangle in this book on spherical triangles but treats it briefly in Book I in much the same way as Lansberg did.[148]

Book V contains such propositions as: "The difference of the sine of two arcs which differ from sixty degrees by the same

145. See p. 103.
146. Pitiscus gives ABC for the triangle, but he uses these same letters for a different purpose in his proof thereby making some of his statements very ambiguous.
147. See p. 95.
148. See p. 97.

amount is equal to the sine of this amount." Pitiscus refers to Finck and Lansberg as also giving this theorem; his proof is the same as the one given by Clavius.[149]

In the 1600 edition the tables of sines, tangents, and secants are computed for a radius equal to 100000 for an interval of a minute.

Disregarding the astronomical section of the book, the problems in the remainder of the text are similar to the one selected for illustration from Clavius' Practica arithmetica.[150]

The fact that the Trigonometria of Pitiscus was reëdited three times and translated into English so soon after its appearance on the Continent indicates the universal esteem with which it was regarded by contemporary mathematicians. Von Braunmühl[151] remarks that in the Dresden library there is a copy of a lecture of Melchior Jöstel[152] entitled "Lectiones in trigonometriam (Bartholomaei) Pitisci. Wittenbergae 1597," which indicates that the Trigonometriae was one of the sources for the lectures in trigonometry that were given in the universities of Germany at the close of the sixteenth century.

In 1613 Pitiscus revised and republished the Thesaurus mathematicus, the table of sines begun by Rheticus and completed after his death by Valentine Otho. Delambre[153] remarks that M. De Prony compared the tables of Rheticus with those published by Pitiscus and drew up a table of the corrections which shows that Rheticus was exact to the fourteenth decimal place, his greatest error being $10^{-16} \times 23$.

13. ADRIAN METIUS

Adrian Metius (1571-1635), professor of mathematics at the University of Franeker, Holland, wrote two brief treatises on plane and spherical trigonometry respectively. The first is included in his Arithmeticae libri duo et Geometriae libri VI which was published at Leyden in 1626 and 1640. The second treatise is Book V of "De astrolabio catholico" which is included in the 1633 edition of Opera omnia astronomica published at Amsterdam.

The plane trigonometry, though brief, is all inclusive. Metius defines the six trigonometric functions with respect to a circle. He states fifteen fundamental trigonometric identities that arise from considering the different cases in a solution of a right triangle, grouping them as problems under three headings called "postulatum 1." etc. He exhibits Regiomontanus' figure and proof for the Sine Law as Axiom 1 of the section devoted to the solution of oblique triangles. In Axiom 2 he gives Finck's figure and proof for the Tangent Theorem, attributes them to Lansberg and adds that Pitiscus gave this theorem very briefly - merely stating it - which is not true, for Pitiscus proved this theorem for a special triangle.[154] The third axiom, the Cosine Theorem, is followed by four problems pertaining to the solution of an oblique triangle. Metius concludes his treatise with an exposition of the trigonometric formula for the area of a triangle and its application to the area of a trapezoid. The tables that follow this treatise are computed by Metius for an interval of ten minutes with a radius equal to 100000.

The treatise on spherical trigonometry is divided into three parts; the first is essentially devoted to spherical geometry, the second exhibits the use of the astrolabe in the solution of right-angled spherical triangles, and the third gives the theorems needed in the solution of triangles by means of the trigonometric tables.

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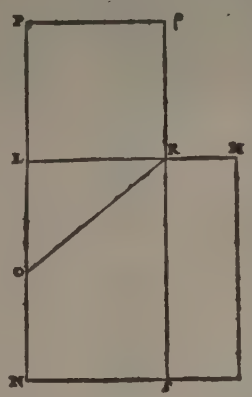
149. See p. 92.
150. See p. 94.
151. Op. cit., p. 226.
152. Jöstel was a professor at the University of Wittenberg and assisted Tycho Brahe for a short time in Prague (Dreyer, op. cit., p. 302).
153. Op. cit., pp. 26-27.
154. See pp. 102-103.



XVIII, cum BKE angulus, duplus sit BAC anguli. Quare hac ratione inuentum est latus XVIII perpendiculũ in part. eius quę ex centro 10000000000. Porrò ei quæ ex centro KB, sit æqualis recta LM. Ad quam describatur MN quadratum. Huius latus LN æqualiter secetur ad O punctum, & per rectam connectantur O, M puncta, & OL continuata directione producatur ad P, fiatq; OL ipsi æqualis rectæ OM. Porrò ad LP describatur quadratum LPQR, & QR producatur ad S. Recta igitur LM media & extrema ratione secta est in puncto R. Sicut enim se habet tota LM ad LR maius segmentum, ita maius segmentum LR ad minus RM. Siquidem SR ad RQ, ita LR ad RM. Subtendens OM potest quantũ vtrunque includentium LM, LO. Illud potest 250000000000000000000000000000: hoc autem 6525000000000000000000000000000. Ergo quadratum ad OM 3125000000000000000000000000000. Atque ipsa OM longitudine erit 559016994374947. At OL part. est 250000000000000. Ergo LP vel LR quarum vtraque æqualis est BE perpendiculo arcus BC in paruo circulo, erit 309016994374947.

AB ea quæ ex centro. 10000000000000000.
Eius Quadratum. 100000000000000000000000000000000.
BK dimidia eius quæ ex centro, vel LM in Quadrato. 50000000000000000.
Eius quadratum. 2500000000000000000000000000000000.
LO dimidia LM. 2500000000000000.
Huius Quadratum. 6250000000000000000000000000000000000.
Quadratum LM. 2500000000000000000000000000000000.
Quadratum OM. 3125000000000000000000000000000.
Longitudo OM. 559016994374947.
Recta LO. 250000000000000.
Recta LP vel LR. 309016994374947. hoc est, BE.

Quoniam igitur recta BE perpendiculum BC arcus part. XVIII exquisitum est, datur & eius basis EA part. 951056516295153, qualiũ AB ea quæ ex centro part. 10000000000000000. Sic quia BF arcum accepimus part. LX, erit BF recta latus hexagoni in circulum inscripti æqualibus tàm lateribus quàm angulis, ac proinde cùm sit ea quæ ex centro BEF circuli, part. 500000000000000, & basis eius 866025403784438. Datur ergo arcus BD part. XXX. Quoniam verò & quatuor laterum figuræ ABEF diametri BF & AE, itemque trila latera BE, AF & AB data sunt, datur per coroll. Quarti Lemmatis lib. Primi EF latus part. 207911690817759. Perpendiculum CD arcus part. XII & FI eius basis part. 978147600733805 per Secund. propos. libri Secundi. Arcu CD ad hunc modum exquisito, dantur per Quartam propos. libri Secundi perpendicula & bases VI part. circuli. Item III & Dodrantis. Similiter & arcus BC quia datus est part. XVIII, dantur per eandem propos. perpendicula & bases IX part. & IIII cum dimidia. Sic & arcus BD cùm inuentus sit part. XXX, dantur etiam XV part. itemque VII & dimidiæ perpendicula & bases sicut ex subiecto calculo manifestum est. Sed arcuum XXXVI & XXIIII part. qui arcuum BC & DC inuentorum dupli sunt, perpendicula dat Tertia propos. proximè antecedentis libri, & bases eorundem arcuum propositio Secunda.

ΔΙΧΟΤΟ-

Basis. 18. part. perpend. 18. part.
Sinus versus. 48541481370484847—48541481370484847.
Quadratum. 2358446197166631381101112914097.
Perpend. 2149110281112562128922212280?.
Quadrat. 9788687740959246966021314648118.
Radix. 312868970080461.
Dimidium. 156434485040230. perpend. 9. part.

Basis. 9. part. perpend. 36. part.
Sinus versus. 9045481340191317
Quadratum. 817179468904863—811116594048163.
Quadrat. Perpend. 9. 1447774181544294001111284152900.
Radix. 116988191415689.
Dimidium. 7849509172.844. perpend. 4. 30.

Basis. 30. part. perpend. 30. part.
Sinus tertius. 866025403784438.
Quadrat. 1133741456211962—1133741456211962.
perpend. 1726931943111288797112709711844
Quadrat. 2567949192411112887797112709711844.
Radix. 5176380902010140.
Dimidium. 2588190451010720. perpend. 15. part.

Basis. 15. part. perpend. 15. part.
Sinus rectus. 258819045102720.
Quadrat. 66982938107782818997110104000.
Perpend. 681483474211863051142126663034.
Radix. 261051518440102.
Dimidium. 130525759220051. perpend. 7. 30.

perpend. 18. part.
3090169943749473—309016994374947.
954915028821154201923377328469.
1564134461040210—1564134461040210.
2447174181242134405111184123900.
10000000000000000000000000000000.
2751218581847177707294448314971000.
Basis. 9. part. 9876883405315117.

7849509172844—7849509172844.
6111815704048504881111288116.
10000000000000000000000000000000.
9914417058716901119781811111664.
Basis. 4. 30. 2969117333333112.

5000000000000—5000000000000.
2500000000000000000000000000000.
2588190451010720—2588190451010720.
669829381077882819971101104000.
10000000000000000000000000000000.
9191017011892219718002638964900.
Basis. 15. 965925826289068.

2590451922210051—2590451922210051.
2703798668514857032000644601.
10000000000000000000000000000000.
9885693911344134277999911773199.
Basis. 7. 30. 9914486173180.

Basis.

LES

TABLES

DES

DIRECTIONS ET PROFECTIONS

DE

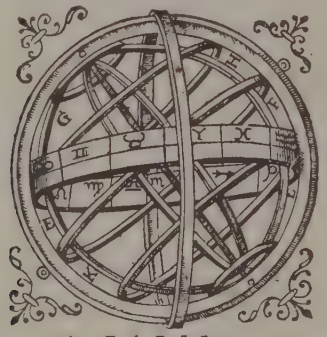
IEAN DE MONT-ROYAL,

CORRIGEES, AVGMENTEES,

ET LEVR VSAGE NON SEVLEMENT

traduit de Latin en François, mais aussi illustré d'annotations, & de figures, tant pour faciliter la practique d'icelles Tables, qu'à les calculer & dresser.

Par D. HENRION, Professeur és Mathematiques.



A PARIS,

En l'Isle du Palais, à l'Image sainct Michel.

M. DC. XXVI.

AVEC PRIVILEGE DV ROY.

ADR. METI ALCMARIANI

ARITHMETICÆ

LIBRI DVO:

ET

GEOMETRIÆ

LIBRI VI.

In quibus etiam Tractatur

TRIGONOMETRIA PLANORVM,

GEODÆSIA,

VSVS CIRCINI & REGVLÆ PROPORTIONALIS,

ARCHITECTVRA MILITARIS,

PROBLEMATA ASTRONOMICA,

SCIATERICA HOROLOGIA.

Editio postrema priore multo auctior.

LVGDVNI BATAVORVM,

Sumptibus Bonaventuræ & Abrahami Elseviriorum,

CIↃ IↃ C EL

tu eadem & meliori ordine, & facilius ostendi posse. Quemadmodum in libello apparet, in quo Ptolomæum carpit. Adiecit his nonnulla Geber, qui nouem libros magnam Ptolomæi cõstructionem conscripsit. Vnde multa sumpserunt Georgius Peurbachius, & Ioannes Regiomontanus, dum prædictam Ptolomaicum opus in epitomen ordinatissimè redigunt. Traditis & complura super his Ioannes prædictus in libellis triangulorum non spernenda. Quæ omnia cum ego proximis his annis vidissem ac contulissem, non passus sum præcepti tanti momenti, & Astronomiæ, pœ planorum triangulorum scientiam, apprimè necessaria sparsim legi. Ea itaque in hos duos libellos congesti adiiciens de ingenij mei riuulo demonstrationes nõnullas. Sic tamen vt post elementa Theodosij ac Menelai demonstrata, detur his locus, vt hæc sint quasi illorum paralipomena. Vnde possint absolui quæ-

tiones, quæ circa sphæralia triangula fieri solent, quæque ad primi mobilis circulos in Astronomia pertinent. Igitur expositis definitionibus, quas negotij fundamentis, veniemus ad demonstrationes, & à facilioribus exorsi ordinem quàm commodissimum seruabimus.

DEFINITIONES.

I. SINVS rectus arcus cuiuspiam est dimidium chordæ dupli talis arcus.

II. Vnde, duorum arcuum qui coniuncti conflant semicirculum idem est sinus, cuius duorum arcuum circulum integrantium eadem est chorda.

III. Et quadrantis sinus est circuli semidiameter: qui sinus totus, siue sinus maximus vocatur. Sicut semicirculi chorda est tota diameter.

circulorum maiorum in superficie sphæræ constituto, cuius versus anguli cuiuslibet ad differentiam duorum sinuum versorũ, quorum vnus est lateris eum angulum subtendentis, alter verò differentiæ duorum arcuum ipsum angulum continentium est sicut quadratũ sinus totius ad id quod sub sinubus arcuum eorundem continetur, rectangulum.

Prop. XXX.

SINVS arcus alicuius ad sinum eius secundum est sicut gnomon ad vmbram rectam eiusdem arcus. Sicut autem sinus secundus ad sinum dicti arcus, hic est gnomon ad vmbram versam ejusdem arcus.

Corollarium.

Vnde vmbra recta cuiusuis arcus est & vmbra versa complementi eiusdem arcus. Item dimidiæ quæ-

dranis debita vmbra tam recta, quam versa æqualis est suo gnomoni. Item gnomon semper est medius proportionalis inter vmbras duorum arcuum quadrantem integrantium seu rectas, seu versas.

Prop. XXXI.

IN triangulo rectangulo ex arcubus circulorum maiorum in superficie sphæræ constituto, sinus totus ad sinum arcus vnius ex acutis angulis subtendentis est sicut quadratũ gnomonis ad rectangulum comprehensum sub vmbris versis, quarum vna complemento reliqui acuti, altera verò arcui eundem acutum subtendenti debetur.

Corollarium.

Hinc manifestum est, quod si in duobus triangulis sphæralibus re-

Plate 35. Adriaan Metius, Leyden, 1640. Plane trigonometry in an elementary arithmetic and geometry. Regiomontanus, Trigonometric tables edited by D. Henrion, Paris, 1626. With solutions of spherical triangles by a new method. Pages from Fr. Maurolico, Paris, 1626. Edition of Menelaus, Sphaera, and Maurolico, On the Sphere.

TRIANGVLORVM GEOMETRIÆ

LIBRI QVATVOR;

In quibus novâ & perſpicuâ methodo, & ἀποδείξει, tota ipſorum Triangulorum doctrina explicatur.

Ad Senatum Populumque Middelburgenſem;

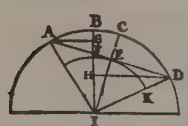
EDITIO SECVNDA

Ab Autore recognita, multiſque in locis aucta.

AMSTERDAMI,
Apud GVILIELMVM BLAEVW,
Anno cIↃ IↃc xxxi.

Vt ſinus anguli ABC 6000000, *ad latus* AC 17: *Ita ſinus anguli* ACB 8823529, *ad latus* AB 25.

15. Si angulorum duorum ſumma detur, quorum ſinuum ratio inter ſe conſtet, ipſi etiam anguli ſecernuntur. Nam ut ſemiſſis ſummæ ſinuum rationis, ad differentiam ſemiſſis, & termini rationis ſinuum alterutrius eſt: ita tangens ſummæ angulorum, ad tangentem anguli; quo minor quæſitus ab angulorum ſummæ ſemiſſe deficit; major eam ſuperat.



Detur in adjecto diagrammate, ſumma angulorum AIB & BID *part.* 40: *cum ratione ſinuum ut* AG *ad* DH (*vel per 4 ſexti elementorum ut* AF *ad* DF) *ut 4 ad 7. Dico utrumque angulum* AIB & BID *ſigillatim datum iri. Egrediatur enim ex* I *recta, biſecans* AD, *rationis ſinuum ſummam datam in* E: *erit* AE 5 1/2, & *angulus* AIE *partium* 20, *æqualis angulo* DIE; FE *vero (differentia termini minoris* AF 4, & AE 5 1/2 *vel* ED 5 1/2 & FD *termini majoris* 7) 1 1/2. *Fiat quoque* IE *radius, ac* DE *tangens ſit anguli* DIE, *vel arcus* KE *partium* 20: *hinc enim dabitur tangens* EF *anguli* EIF *vel* BIC *ſubtendens. Nam per 19 ſeptimi Euclidis*

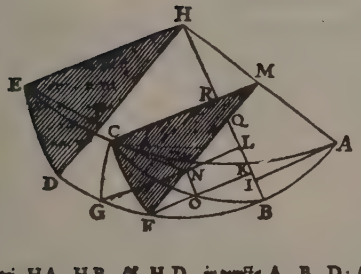
Vt DE 5 1/2, *ad* DE *tangentem ang.* DIE 3639702: *Ita* FE 1 1/2, *ad* FE *tang. anguli* EIF, 992646

Cujus arcus è Tangentium Canone datur partium 5 40′ 8″ *fere. Atqui hic angulo major eſt angulus* BID *major: minor vero angulus* AIB *minor. Ergo* BID *angulus eſt partium* 25 40′ 8″ *fere:* AIB *partium* 14 19′ 52″: *Quod erat demonſtrandum.*

ΠΟΡΙΣΜΑ.

Itaque duobus obliquanguli Trianguli lateribus datis, & angulo ab iis comprehenſo, inveniuntur anguli reliqui, & latus tertium. Nam ut ſemiſſis ſummæ laterum datorum, ad differentiam ſummæ ſemiſſis, & lateris alterutrius: Sic tangens ſemiſſis reſidui anguli ad ſemiculum, ad Tangentem anguli, quo angulus minori lateri oppoſitus eâdem ſemiſſe minor, ma-

 jori



lis erit arcus BC; *& proinde arcus* BF *differentia erit laterum* AC *& AB, & arcus* GF *differentia tertii lateris* BC, *& reliquorum laterum differentia* BF. *Emittatur deinde ex* H *communi centro quadrantum* AD *&* AF, *ſemidiametri* HA, HB, *&* HD, *in puncta* A, B, D; *& à terminis arcuum* AB, BF, *&* BG, *demittantur perpendiculares* AI, FK, *&* GL, *in ſemidiametrum* HB; *erunt hæ arcuum dictorum rectis ſinus, per 7 primi hujus;* BI *autem,* BK, *&* BL, *verſi ſinus eorundem per 10 ejuſdem: & proinde* KL *differentia ſummæ verſorum lateris* BC *vel* BG, *& reliquorum laterum differentia* BF. *Præterea à termino arcus* AF *deſcendat perpendicularis* FM *in ſemidiametrum* HA, *erit hæc ſinus rectus lateris* AF. *Vbi autem* GL *&* FM *ſeſe interſecant ſit* N *punctum, ex quo ducatur* NO *parallela* HB; *adeoque per 34 primi elementorum æqualis ipſi* KL. *Adhæc à termino arcus* DE, *demittatur perpendicularis* EP *in ſemidiametrum* HD, *erit hæc ſinus rectus arcus* DE; *&* DP *ſinus verſus ejuſdem. Poſtremò à* C *communi termino arcuum* FC *&* GC *ducatur recta* CN *in* N, *ſectionem rectarum* GL *&* FM; *erit hæc normalis rectæ* GN *&* FN. *Arcus enim* FC *&* GC *per 5 hujus normales ſunt quadrati* ABD, *tranſeuntis per* A *&* B *polos eorundem. Itaque communis eorum ſectio, quæ per 3 undecimi Euclidis eſt recta linea, nempe* CN, *eſt plano quadrantis* ABD *normalis, per 19 ejuſdem. Tranſit autem ſectio communis arcuum dictorum per* N *punctum, ex converſione definitionis lineæ perpendicularis ſuper planum rectæ. Quare* CN *eſt ſinus rectus arcus* FC, *&* FN *ſinus verſus ejuſdem.*

His verò in hunc modum expeditis, dico DH *radium eſſe ad* FM *ſinum rectum lateris* AC, *ut* DP *ſinus verſus anguli ad* A, *ad* FN, *ſinum verſum arcus* FC. *Item* HA *radium eſſe ad* AI, *ſinum rectum lateris*

 AB,

TANGENTIVM & SECANTIVM.

30	Sinus		Tangens		Secans		
30	50753.84	86162.92	58904.50	169766.31	116059.21	197029.44	30
31	50778.90	86148.15	58943.69	169653.44	116079.11	196932.20	29
32	50803.96	86133.37	58982.89	169540.69	116099.02	196835.07	28
33	50829.01	86118.59	59022.11	169428.04	116118.95	196738.05	27
34	50854.06	86103.80	59061.34	169315.50	116138.89	196641.14	26
35	50879.10	86089.00	59100.58	169203.08	116158.85	196544.34	25
36	50904.14	86074.20	59139.83	169090.77	116178.83	196447.67	24
37	50929.18	86059.39	59179.10	168978.56	116198.82	196351.10	23
38	50954.21	86044.57	59218.39	168866.47	116218.83	196254.64	22
39	50979.24	86029.74	59257.68	168754.49	116238.86	196158.29	21
40	51004.26	86014.91	59296.99	168642.61	116258.91	196062.06	20
41	51029.28	86000.07	59336.32	168530.85	116278.97	195965.93	19
42	51054.29	85985.22	59375.66	168419.19	116299.05	195869.92	18
43	51079.30	85970.37	59415.01	168307.65	116319.14	195774.01	17
44	51104.31	85955.51	59454.37	168196.21	116339.25	195678.22	16
45	51129.31	85940.64	59493.75	168084.89	116359.38	195582.54	15
46	51154.31	85925.76	59533.14	167973.67	116379.53	195486.97	14
47	51179.30	85910.88	59572.54	167862.56	116399.69	195391.50	13
48	51204.29	85895.99	59611.96	167751.56	116419.87	195296.15	12
49	51229.27	85881.09	59651.40	167640.67	116440.07	195200.91	11
50	51254.25	85866.18	59690.84	167529.88	116460.28	195105.77	10
51	51279.22	85851.27	59730.30	167419.21	116480.51	195010.75	9
52	51304.19	85836.35	59769.78	167308.64	116500.76	194915.83	8
53	51329.16	85821.42	59809.27	167198.18	116521.02	194821.02	7
54	51354.12	85806.49	59848.77	167087.82	116541.30	194726.32	6
55	51379.08	85791.55	59888.28	166977.58	116561.60	194631.73	5
56	51404.04	85776.60	59927.81	166867.44	116581.91	194537.25	4
57	51428.99	85761.64	59967.35	166757.41	116602.24	194442.88	3
58	51453.93	85746.68	60006.91	166647.48	116622.59	194348.61	2
59	51478.87	85731.71	60046.48	166537.66	116642.96	194254.45	1
60	51503.81	85716.73	60086.06	166427.95	116663.34	194160.40	0

Plate 36. Philip Lansberg, Triangulorum Geometriae libri quatuor, 1631.

Mr. BLUNDEVIL
HIS

EXERCISES,

CONTAYNING

Eight Treatiſes,

The Titles whereof are ſet down in the next printed page.

Which Treatiſes are very neceſſary to be read and learned of all young Gentlemen, that have not been exerciſed in ſuch Diſciplines, and yet are deſirous to have knowledge as well in *Coſmographie*, *Aſtronomie*, and *Geographie*, as alſo in the Art of *Navigation*, in which Art it is impoſſible to profit without the help of theſe or ſuch like Inſtructions.

To the furtherance of which Art of *Navigation*, the ſaid Mr. BLUNDEVIL ſpecially wrote the ſaid Treatiſes, and of meere good will doth Dedicate the ſame to all young Gentlemen of this Realme.

The ſeventh Edition corrected and ſomewhat enlarged by Ro. Hartwell *Philomathematicus.*

LONDON,
Printed by *Richard Biſhop*, and are to be ſold by *Benjamin Allen* at the ſigne of the Flowerdeluce in Popes-head Alley. 1636.

To finde the proportionall parts when the Sine is not preciſely found in the Tables.

If you ſhall not finde in the Canon, the Sine which by your calculation is found; for that you may take the Sine next leſſer or greater, (for example ſake) if by calculation you finde the Sine 50786, conferre that with the Sines 50779 and 50804, put in the Canon, by ſubtracting the one from the other thus;

50786	50804
50779	50786
7	18

And ſeeing you perceive that the Sine 50786 to differ leſſe from the Sine 50879 than from the Sine 50804, I ſay the arke or angle 30 *d.* 31′ agreeth to that, which arke or angle in the Canon you ſhall perceive to bee aſcribed to the *Sine* 50779.

The ſame reaſon is to be obſerved in the uſe of the Tables of Tangents and Secants.

Now if you deſire to proceed to the proportionall part, and to have ſecond ſcruples, Then ſubtract the leſſer tabulary Sine from the greater, and the remaynder ſhall bee the firſt number in the rule of proportion, and 60 ſecounds the ſecond number, the former leſſer difference the third, being that found by calculation.

50786	50804	50804
50779	50786	50779
7	18	25

25 —— 60″ —— 7
7
420

(2
XZ
ΛΖ (0(16″
ΣΣΣ
Z

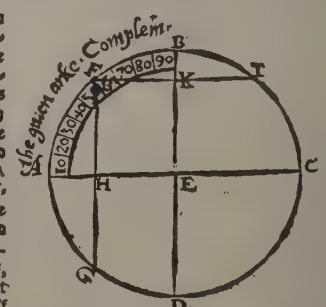
THE TABLE OF TANGENTS FOR EVERY MINVTE OF THE QVADRANT.

First calculated
By I. REGIO MONTANVS.

But lately examined, and in many places corrected and amended according to BARTHO. PITISCVS.

ſometime make the totall Sine to be but 100000, ſo as you cut off the two laſt figures on the right hand in every Sine, but you ſhall better underſtand every thing here above mentioned, by the figure demonſtrative here following.

The Figure Demonſtrative.



In this figure you ſee firſt a whole Circle drawne upon the Center E, and marked with the letters A, B, C, D, which Circle by two croſſe Diameters, marked with the letters A C, and B D, and paſſing both through the Center E, is divided into foure Quadrants or Quarters, the upper Quadrant whereof, on the left hand is marked with the letters A, B, E, in which Quadrant the right perpendicular line marked with the letters F, H, betokeneth the right Sine of the given Arke A F, which right Sine is the one halfe of the Chord or String F G, and the given Arke A F, is the one halfe of the double Arke or Bow G A F, and A H is the Shaft, called in Latine Sinus verſus: Againe, the letters F B do ſhew the complement, which together with the given Arke A F, do make the whole Quadrant A F B, which is divided into 9 ſpaces, every ſpace containing 10 degrees, whereby you may plainly perceive, that in this demonſtration, the given Arke A F, is 50 degrees, and the complement F B, is 40 degrees, both which being added together, do make up the whole Quadrant of

With the geometrical definitions of the first part Metius demonstrates the polar triangle in a manner similar to that of Lansberg.[155] In parts two and three he marks the unknown sides of a triangle with one stroke. The thirty-two trigonometric identities used in the solution of spherical right triangles, as given by Magini[156] are listed without proofs or figures in part three. Metius solves oblique triangles by resolving them into right triangles; he uses the term "casus," just as Regiomontanus did, for the portions of the base made by the perpendicular drawn to it from the opposite vertex. He also solves oblique triangles by applying the sine, cosine, and tangent theorems. He adds an extra chapter on the solution of plane right triangles. The tables of sines, tangents, and secants that follow this treatise are computed for a radius equal to 100000 for an interval of one minute.

14. TYCHO BRAHE AND MELCHIOR JÖSTEL

In the first volume of his Tychonis Brahe Dani Opera omnia[157] Dreyer has edited two manuscripts pertaining to trigonometry. The one, Triangulorum planorum et sphaericorum praxis arithmetica was written by Tycho Brahe in 1591 and the other, Problemata de triangulis dated 1599, is the work of Melchior Jöstel.

Tycho Brahe's treatise consists of sixteen "dogmas"; the first three concern the solution of plane right triangles, the next four, the solution of plane oblique triangles, and the remaining nine deal with spherical triangles.

Tycho Brahe does not prove but merely states how the triangles should be solved. His treatise shows his familiarity with the work of Vieta. His version of the tangent theorem will be given as an illustration of his other "dogmas." He uses it in the solution of an oblique triangle when two sides and the included angle are given. His method is as follows:

I. Given an angle (acute or obtuse), take the complement of it from a semicircle, i.e., 180, take half of this and find the tangent of it in the table of tangents.

II. Add the smaller of the two given sides to the larger and take one-half of this sum; this will be "inventum I."

III. From "inventum I." subtract the smaller of the two given sides; this will be "inventum II."

IV. The third "inventum" will be the tangent of the angle which is half the complement of the given angle.

Rule

Multiply "inventum II" by "inventum III" and divide by "inventum I." This will give the tangent of one-half the difference of the angles which are opposite the two given sides. If this angle is found in the table of tangents and subtracted from half the complement of the given angle it will be the angle opposite the smaller of the two given sides.

Jöstel's treatise consists of two problems relating to spherical triangles, the statement of the Sine Law for plane oblique triangles, and a problem pertaining to the solution of a plane right triangle. The two problems in spherical trigonometry consider respectively the determination of the sides of a triangle when the angles are given and the evaluation of the third side and the remaining angles when two sides and the included angle are given. The problem in plane trigonometry pertains to the solution of a plane right triangle when two sides which include the right angle are given. The several figures in the treatise have the given parts marked with a stroke and the required parts with a small circle.

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155. See p. 97.
156. See p. 100.
157. This work was published in Hveen in 1913.

15. NATHANIEL TORPORLEY

The trigonometry of Nathaniel Torporley (1563-1632), who is said to have been secretary to Vieta and conveyed Vieta's ideas to Harriot[158] in England,[159] is embodied in his Diclides coelometricae seu valvae astronomicae universales, which was published in London in 1602 and which is the most unique book ever written on trigonometry.

If Torporley had not disguised his trigonometrical notions under so many metaphors, credit for the invention of "circular parts" which is now universally given to Napier might be ascribed to Torporley, as A. de Morgan[160] has shown, for in a most ingenious but obscure manner Torporley has reduced the six cases of right spherical triangles to two.

Just what constitutes the trigonometric portion of the Diclides coelometricae[161] is explicitly stated in the fifth paragraph of an introductory article entitled "Opusculi levis adumbratio inscriptioque" which precedes the preface of the Diclides:

> The easiest treatise on the doctrine of triangles, which reduces the general system to six diagrams which are easily arranged on a figure of a mitre is given in the second part of the following; whence is furnished all the necessary data which will be needed for the use of the author's tables or the canon of triangles, without preserving the perplexing precepts included in the works of others.

In the text proper Torporley compares the given parts in the six cases of a right triangle in the following manner:

(1) Two sides and an angle opposite one of them is a crow.

(2) Two angles and the hypotenuse is a spear.

(3) Hypotenuse, side and angle included by them is a shears.

(4) Hypotenuse, side and angle opposite is a siphon.

(5) Three sides is a prison.

(6) Two angles and a side is a sling.

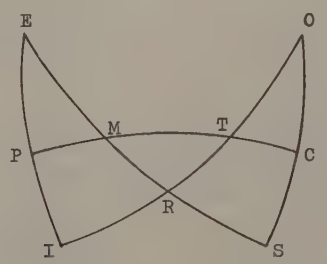

Fig. 131

The mitre is constructed as follows:

E is the pole of OTRI, a portion of a great circle which is less than a semicircle.

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158. Thomas Harriot is the English mathematician who was sent in 1585 to survey Virginia, now North Carolina.

159. Florian Cajori, op. cit., p. 152.

160. "On the Invention of Circular Parts," Philosophical Magazine, 23:352, 1843.

161. A fair description of the contents of this book is given by Delambre, Histoire de l'Astronomie Moderne (Paris: Courcier, 1821), vol. II, pp. 37-40.

sit, æquatur quadrato adgregati ex base primi, & perpendiculo secundi, plus quadrato differentiæ inter perpendiculum primi & basin secundi, vel etiam, æquatur quadrato adgregati ex perpendiculo primi & base secundi, plus quadrato differentiæ inter basin primi & perpendiculum secundi.

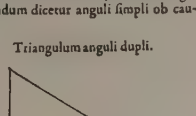
PROPOSITIO XLVIII.

A Duobus triangulis rectangulis æqualibus & æquiangulis, tertium triangulum rectangulum, constituere.

Sunto duo triangula rectangula, quorum communia latera hypotenusa quidem A, perpendiculum B, basis D. Oporteat ab illis duobus tertium triangulum rectangulum constituere. Fiat deductio sicut docuit Propositio 46. casu primo. Deduci enim tantum potest synæreseos via, non autem diæreseos. Fit hypotenusa similis A quadrato. Basis, D quadrato = B quadrato. Perpendiculum B in D 2. Tertium autem illud, vocetur triangulum anguli dupli. & ejus respectu primum vel secundum dicetur anguli simpli ob causas * suo ponendas loco.

* *Causa est quod angulus acutus trianguli rectanguli à duobus triangulis via synæreseos effecti æquetur angulo acuto horum triangulorum simul adgregato, cujus theorematis conversum demonstravit Andersonus theoremate secundo sectionum angularium. Porro acuti voce intelligitur is angulus, cui Perpendiculum subtenditur.*

Triangulum anguli dupli.



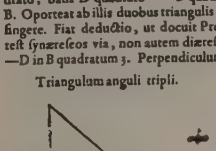
Din.B 2

PROPOSITIO XLIX.

A Triangulo rectangulo simpli, & triangulo rectangulo anguli dupli, triangulum rectangulum effingere. Vocetur autem tertium illud, triangulum anguli tripli.

Sunto duo triangula rectangula, unum anguli simpli, cujus hypotenusa A, perpendiculum B, basis D. Alterum anguli dupli, cujus consequenter hypotenusa sit similis A quadrato, basis D quadrato = B quadrato, perpendiculum simile plano duplo, ex D in B. Oporteat ab illis duobus triangulis rectangulis, tertium triangulum rectangulum effingere. Fiat deductio, ut docuit Propositio 46. primo casu. Effingi enim tantum potest synæreseos via, non autem diæreseos. Fit hypotenusa, A cubus. Basis, D cubus — D in B quadratum 3. Perpendiculum, D quad. in B 3 — B cubo.

Triangulum anguli tripli.



PROPOSITIO L.

A Triangulo rectangulo anguli simpli, & triangulo rectangulo anguli tripli, tertium triangulum rectangulum constituere. Vocetur autem illud, triangulum anguli quadrupli.

Sunto duo triangula rectangula. Unum anguli simpli, cujus hypotenusa A, perpendiculum B, basis D. Alterum anguli tripli, cujus hypotenusa consequenter sit similis, A cubo. Basis, Dc, — D in Bq. 3. Perpendiculum, simile D q. in B 3, — Bc. Oporteat ab illis tertium triangulum rectangulum constituere. Fiat deductio, ut docuit Propositio 46. casu primo. Fit hypotenusa similis, A qq. Basis Dqq. — Dq. in Bq. 6, + Bqq. Perpendiculum B in Dc 4. — Bc in D 4.

— 1 C. aquabitur 20. & sunt duo latera, a quibus quadrata adjuncta rectangulo à lateribus, faciunt 21. aggregatum autem laterum, ductum in rectangulum est 20. & sit 1 N in aequalitate inverse negata, alterutrum è lateribus, maius minuve; in directe vero negata, aggregatum ipsorum laterum, ex hypothesi laterum. 1. & 4.

Neque vero in Geometrica phrasi hic erit magna dissimilitudo: Enimvero dicet Geometra, Bplanum esse aggregatum quadratorum à tribus proportionalibus lineis rectis, D vero solidum quod fit ab aggregato extremarum in mediæ quadratum, seu ab alterutra extrema in aggregatum quadratorum à reliquis, & fieri A aggregatum extremarum, E vero primam vel tertiam.

Sic in exposito themate, 21. est aggregatum quadratorum à tribus proportionalibus, & solidum 20. est ab aggregato extremarum in medias quadratum, seu ab alterutra extrema in quadrata à reliquis ex serie proportionalium 1. 2. 4.

Ac elegantius & præstantius ex analyticis angularium sectionum hujusmodi æqualitatum constitutio eruitur, & ad Δ*ιοφαντικὰ* accommodatius in hanc formulam:

ALITER,

TERTIVM THEOREMA.

SI A cubus — B quad. 3 in A, æquetur B quad. in D, sit autem B major D semisse: B quad. 3 in E, — E cubo æquabitur B quad in D.

Et sunt duo triangula rectangula æqualis B hypotenusæ, ita ut angulus acutus subtensus à perpendiculo primi, sit triplus ad angulum acutum subtensum à perpendiculo secundi: basis vero dupla primi, est D, & sit A dupla basis secundi. E vero basis simpla secundi, contracta, protractave longitudine ejus quæ potest quadrato triplum perpendiculi ejusdem.

1 C — 300 N. *aequetur* 432. *vel etiam* 300 N. — 1 C *aequetur* 432. *sunt duo triangula rectangula. quorum hypotenusa communis est* 10: *ita ut angulus acutus primi, à perpendiculo videlicet subtensus. sit triplus ad acutum secundi, à suo quoque perpendiculo subtensum; basis autem primi dupla, est* $\frac{600}{100}$. *& 1 N in aequalitate directe negata est basis dupla secundi: in inverse vero negata, est basis simpla secundi. plus minusve ea quæ potest quadrato triplum perpendiculum secundi.*

Constituta hypotenusa communi 10. *basi secunda trianguli* 9. *sit perpendiculum ejusdem secundi* $\sqrt{19}$.

Primi vero hypotenusa stante 10. *sit basis* $2\frac{16}{100}$. *itaque quem in ea hypothesi dicetur* 1 C — 300 N. *aquari* 432. *fiet* 1 N 18. *vel quum dicetur* 300 N. — 1 C *aquari* 432. *Fiet* 1 N. 9 + $\sqrt{17}$. *vel* 9 — $\sqrt{17}$.

Atque hæc ad assequendum ex Zeteticis constitutiones adæquatarum quæ affectæ sunt potestatum, exempla nunc sufficiuntor. Etsi enim de simpliciter affectis Theoremata tantum proposita sunt, tamen quemadmodum ad multipliciter affectas possint trahi, vix ignorabitur, quando præsertim detegetur earum plasma, de quo jam succedit dicendi locus. Ipsorum enim Zeteticorum à quibus antedictæ constitutiones depromptæ sunt, jam patuit processus sufficienter in expositis eclogis, hoc postremo Theoremate excepto, quod par est rejici in peculiarem Zeteticorum ad angulares sectiones pertinentium silvulam.

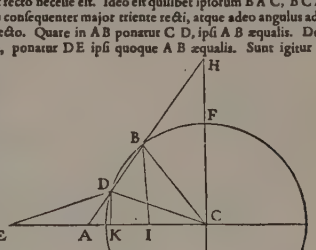
CAPVT VII.

De generali methodo transmutandarum æquationum.

PLasmatis ratio pendet è doctrina transmutandarum æquationum generaliter imprimis proponenda & demonstranda.

PROPOSITIO XVII.

Si duo triangula fuerint æquicrura singula, & ipsa alterumalteri cruribus æqualia, angulus autem, quem is qui est ad basin secundi relinquit è duobus rectis, triplus est ipsius anguli B A C, & major sit recto necesse est. Ideo est quilibet ipsorum B A C, B C A qui sunt ad basin angulorum consequenter major triente recti, atque adeo angulus ad verticem A B C est minor recto. Quare in A B ponatur C D, ipsi A B æqualis. Deinde ex D in C A productum, ponatur D E ipsi quoque A B æqualis. Sunt igitur duo triangula æquicrura A B C, E D C, & ipsis AB, B C cruribus æqualibus primi trianguli, æqualia sunt D E, D C crura secundi trianguli. Qualium autem utreque angulorum B A C, B C A est pars una, angulus A B C est duorum rectorum minus duabus talibus partibus, & angulus exterior anguli A B C duarum est illarum partium, cui angulo exteriori æquatur angulus A D C: quoniam sunt æquales anguli D B C, C D B ob æqualitatem quoque crurum C B, C D. Ex angulo autem A D C & D A C, compositus est angulus exterior anguli D C A. Secundum itaque triangulum est C D E ipsum æquicrurum, & crura habens æqualia cruribus primi A B C, & utrumque angulorum quem anguli ad basin D C E vel D E C relinquunt è duobus rectis, triplum anguli B A C vel B C A. Dico igitur triplum solidum sub A C & quadrato ex A B minus cubo ex A C, æquari solido sub E C & quadrato ex D C seu A B.



Centro enim C intervallo C B vel C D describatur circulus, & agatur diametet F C G secans E A, uti continuatur, perpendiculariter in C, ipsam vero A D quoque continuatam in H, ipsi quoque F G agantur parallelæ B I, D K, secantes eandem A E perpendiculariter in I, K. Sunt igitur A I, I C æquales, & ideo fit A C ipsius A I dupla. Itaque sunt quoque æquales A B, B H, & fit A H ipsius A B dupla. Æquales quoque E K, K C, & fit E C dupla ipsius E K.

Quadratum autem ex C H, æquale est quadrato ex C F, id est A B una cum eo quod fit sub H F, H G, & convertendo quadratum ex C H minus quadrato ex A B, æquale est ei quod fit sub H F, H G, hoc est ei quod fit sub H B, H D. Ipsum porro quadratum ex C H, æquale est quadrato ex A H minus quadrato ex A C. & quadratum ex A H est quadruplum quadrati ex A B. Triplum igitur quadratum ex A B minus quadrato ex A C, æquale est ei quod fit sub B H, H D. Sed est H B ad H D, sicut C I ad C K, & est C I ad C K, sicut C A ad C E: cum sint hæ illarum duplæ. Quare sicut C A ad C E, ita H B ad H D, & consequenter sicut C A ad C E, ita quadratum ex H B, id est A B ad id quod fit sub H B, H D, id est, ad triplum quadratum ex A B, minus quadrato

Plate 38. Pages from Franciscus Vieta, Opera Omnia, Leyden, 1646. Published by the Elzevir press.

AD

ANGVLARES SECTIONES

THEOREMATA ΚΑΘΟΛΙΚΩΤΕΡΑ DEMONSTRATA

PER

ALEXANDRUM ANDERSONUM.

THEOREMA I.

SI fuerint tria triangula rectangula, quorum primi angulus acutus, differat ab acuto fecundi, per acutum tertii, & fit exceſſus penes primum, latera tertii recipiunt hanc ſimilitudinem.

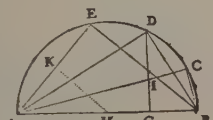
Hypotenuſa, fit ſimilis rectangulo ſub hypotenuſis primi & ſecundi.

Perpendiculum, ſimile rectangulo ſub perpendiculo primi & baſe ſecundi, minus rectangulo ſub perpendiculo ſecundi, & baſe primi.

Baſis, rectangulo, ſub baſibus primi & ſecundi, plus rectangulo ſub perpendiculis eorundem.

Is angulus acutus intelligitur, cui latus perpendiculi voce deſignatum ſubtenditur. reliquum à recto, cui baſi. hypotenuſa vero, latus recto ſubtenſum.

Sint tria triangula AEB, ADB, ACB, quorum baſes ſint AE, AD, AC, perpendicula EB, DB, CB. ſecet E B baſin AC in I puncto : & demittatur perpendicularis DG in rectam A B.



Eritque ut AD ad DB, ita AE ad EI, id eſt EB minus IB. & rectangulum DB in AE, æquale erit rectangulo AD in EB minus AD in IB, additoque communi AD in IB : DB in AE plus AD in IB, æquabitur AD in EB. eſt quoque ut AD ad AB, ita CB ad IB, & rectangula A B in CB, AD in IB æqualia : ergo DB in AE plus AB in BC, æqualia erunt ipſi AD in EB. & ablato communi (AE in DB,) AB in BC æquabitur AD in EB minus AE in DB, quibus ipſi AB applicatis orietur latitudo BC, eritque AB quadratum ad AD in EB minus AE in DB. ut AB ad BC. Quod erat demonſtrandum.

Rurſus ut AG ad AD, ita AE ad AI, & AD in AE, æquale erit ipſi AG in AI; eſt autem AB in AC, æquale AG in AI plus AG in IC plus GB in AC; & GB in AC, æquale ipſis GB in AI plus GB in IC. eſt quoque AG in IC plus GB in IC, æquale AB in IC, & proinde AG in AI plus AB in IC plus GB in AI, æqualia erunt AB in AC : ſed GB in AI,

$$\frac{B\,qc. - Z\,q. \text{ in } B\,c5. + Z\,qq. \text{ in } B\,5.}{Z\,qq.}$$ æquale erit ſextæ.

$$\frac{B\,qqq. - Z\,q. \text{ in } B\,qq.\,6. + Z\,qq. \text{ in } B\,q.\,9. - Z\,qqq.\,2.}{Z\,qc.}$$ ſeptimæ.

$$\frac{B\,qqc. - Z\,q. \text{ in } B\,qc.\,7. + Z\,qq. \text{ in } B\,c.\,14. - Z\,qqq. \text{ in } B\,7.}{Z\,cc.}$$ octavæ.

$$\frac{B\,qcc. - Z\,q. \text{ in } B\,cc.\,8. + Z\,qq. \text{ in } B\,qq.\,20. - Z\,cc. \text{ in } B\,q.\,16. + Z\,qcc.\,2.}{Z\,qqc.}$$ nonæ.

$$\frac{B\,ccc. - Z\,q \text{ in. } B\,qqc.\,9. + Z\,qq. \text{ in } B\,qc.\,27. - Z\,qqq. \text{ in } Bc.\,30. + Z.\,qqqq. \text{ in } B\,9.}{Z\,qcc.}$$ decimæ.

Atque ita deinceps. Quod erat demonſtrandum.

In notis, ſit ſemidiameter 1. baſis prima 1 N. Erit

1 Q — 2	
1 C — 3 N	
1 QQ — 4 Q + 2	Duplı.
1 QC — 5 C + 5 N	Tripli.
1 CC — 6 QQ + 9 Q — 2	Baſium Quadrupli.
1 QQC — 7 QC + 14 C — 7 N	Quintupli.
1 QCC — 8 CC + 20 QQ — 16 Q + 2	Sextupli.
1 CCC — 9 QQC + 27 QC — 30 C + 9 N	Septupli.
	Octupli.
	Nonupli.

Et ſic continuo radicem binarium cum ſibi proximo jungendo, & compoſitum cum numero illi proximè deinceps componendo, creabuntur reliqui efficientium multiplicium numeri in infinitum, juxta ſerium ſubjecta tabella.

NUMERI MULTIPLICIUM ADFECTIONIS.

Prima Negata.	Secunda affirmata.	Tertia negata.	Quarta affirmata.	Quinta negata.	Sexta affirmata.	Septima negata.	Octava affirmata.	Nona negata.
2								
3								
4	2							
5	5							
6	9	2						
7	14	7						
8	20	16	2					
9	27	30	9					
10	35	50	25	2				
11	44	77	55	11				
12	54	112	105	36	2			
13	65	156	182	91	13			
14	77	210	294	196	49	2		
15	90	275	450	318	140	15		
16	104	352	660	672	336	64	2	
17	119	442	935	1122	714	204	17	
18	135	546	1287	1782	1386	540	81	2
19	152	665	1729	2717	2508	1254	287	19
20	170	800	2275	4604	4290	2640	825	100
21	189	952	2940	5733	7007	5148	2079	385

& tranſſinuoſa tertia, ita quod ſit ſub tranſſinuoſa complementi ſecunda & ſinus complementi tertia ad id quod ſit ſub ſinu toto & ſinu complementi primæ.

Et,

16 *Vt rectangulum quod ſit ſub ſinu toto & profinu primæ ad id quod ſit ſub ſinu ſecundæ & profinu tertiæ. ita quod ſit ſub tranſſinuoſa complementi ſecundæ & profinu complementi tertiæ ad id quod ſit ſub ſinu toto & profinu complementi primæ.*

DATI SEXTI.

Πεφασμένη I.

Peripheriam intellige tripleure ſphæricæ conſtituendi idoneam. Itaque ne ſemicirculum excedante.

Data peripheria compoſita è duabus peripheriis, quarum tranſſinuoſis datam habeant rationem, dantur ſingulæ.

1 *Enimvero ſi compoſita minor eſt circuli quadrante*,

Erit,

Vt tranſſinuoſa componentium prima ad tranſſinuoſam ſecundæ, ita tranſſinuoſa complementi compoſita ad profinum ſecundæ. plus profinu complementi compoſitæ.

2 *Et ſi compoſita major eſt quadrante circuli, utraque vero componentium minor quadrante*,

Erit,

Vt tranſſinuoſa primæ ad tranſſinuoſam ſecundæ, ita tranſſinuoſa complementi compoſitæ ad profinum ſecundæ minus profinu complementi compoſitæ.

3 *Et ſi denique componentium peripheriarum prior ſit minor quadrante, ſecunda major*,

Erit,

Vt tranſſinuoſa primæ ad tranſſinuoſam ſecundæ, ita tranſſinuoſa complementi compoſitæ ad profinum complementi compoſitæ minus profinu ſecundæ.

Et,

Vt tranſſinuoſa ſecundæ ad tranſſinuoſam primæ, ita tranſſinuoſa complementi compoſitæ ad profinum complementi compoſitæ plus profinu ſecundæ.

II.

Data differentia duarum peripheriarum, quarum tranſſinuoſæ datam habeant rationem, dantur ſingulæ.

1 *Enimvero ſi differentia ſit major circuli quadrante*,

Erit,

Vt tranſſinuoſa differentiæ primæ ad tranſſinuoſam ſecundæ, ita tranſſinuoſa complementi differentiæ ad profinum complementi differentiæ plus profinu ſecundæ.

2 *Et ſi differentia ſit minor quadrante, differentes autem peripheriæ diverſæ ſint ſpeciei*,

Erit,

Vt tranſſinuoſa primæ ad tranſſinuoſam ſecundæ, ita tranſſinuoſa complementi differentiæ ad profinum ſecundæ minus profinu complementi differentiæ.

3 *Et ſi denique differentia ſit minor quadrante, utraque vero differentium vel minor quadrante, vel utraque major. Prima autem intelligatur ea ad quam pertinet tranſſinuoſa major*,

Erit,

Vt tranſſinuoſa primæ ad tranſſinuoſam ſecundæ, ita tranſſinuoſa complementi differentiæ ad profinum ſecundæ minus profinu complementi differentiæ.

Et,

Vt tranſſinuoſa ſecundæ ad tranſſinuoſam primæ, ita tranſſinuoſa complementi differentiæ ad profinum primæ minus profinu complementi differentiæ.

III.

ALITER.

Data ſumma vel differentia duarum peripheriarum, quarum tranſſinuoſæ datam habeant rationem, dantur ſingulæ.

1 *Enimvero ſi utraque peripheria propenatur minor quadrante vel utraque major.*

Plate 39. Pages from Franciscus Vieta, Opera Omnia, Leyden, 1646. Published by the Elzevir press.

O is the pole of EMRS which intersects OTRI in right angles at R.

IRT equals OTR equals a quadrant.

T is the pole of IPE which is a quadrant.

M is the pole of SCO which is a quadrant.

SRM equals RME equals a quadrant.

PT equals MC equals a quadrant; the angles at P and C are right angles.

If triangle MRT has the properties of (1), i.e., MR, TR and M are known, then the following is true:

RT = 90 - OT, hence TO is known.
MR = 90 - RS, hence RS is known, hence angle O is known.
SC measures angle M, hence SC is known, but SC = 90 - OC, so OC is known.

Therefore triangle TOC has the properties of (3).

Similarly triangle PEM has the properties of (2).

If triangle MTR were characterized as (4) then triangles PEM and TOC would have the properties of (5) and (6) respectively.

Once Torporley's trigonometric method is established, his probable imitation of Bressieu can be detected, for in Propositions 29 and 30 of Book IV in his Metricis astronomiae Bressieu applies to Fig. 132 the two following rules:

In a spherical right triangle ABC

$$\frac{\text{cos of a side}}{\text{cos of angle opposite}} = \frac{1}{\text{sine of other angle}}$$

$$\frac{\text{tan of an angle}}{\text{cot of other angle}} = \frac{1}{\text{cot of hypotenuse}}$$

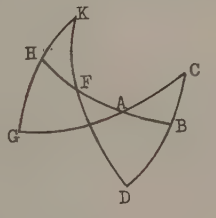


Fig. 132

It should be recalled that in several of his propositions Bressieu considered the quadrilateral and applied the theorem of Menelaus.[162]

Torporley's treatise has two tables, the first of which is the same as the "Tabula tertiae analogiae" later published by Magini.[163] This table is to be used when the triangle MTR has the properties of (1). If the two sides MR and TR are respectively 23° and 19°48' and angle M is 58°, then tan 19°48' = sin 23 tan 58°.

	58	
23	19	48
24	20	41

If MRT is characterized by (4), then the second table should be used. The hypotenuse MT is 54°, side TR is 30°46' and angle M opposite TR is 39°14', whence sin 54° sin 39°14' - sin 30°46'.

Delambre[164] mentions that Torporley is obscure; however, his two tables show many of the benefits of the device of Napier.

	70			
	G	G	M	M
	39		14	
54		30		46
	8		26	

16. SIMON STEVIN

The first volume of Hypomnemata mathematica, Snellius' translation of Stevin's Wiskonstige Gedachtenissen, both published in 1608, contains Stevin's contribution to

162. See p. 88.
163. See p. 98.
164. Loc. cit.

trigonometry under the title "Cosmographie doctrinae triangulorum." In this treatise Stevin uses the term "sinus-sagitta" or "sagitta" for the vers sine. His notation for minutes and seconds is similar to that which he used for decimal fractions; for instance, he writes 45'22" as 45 ① 22 ② .

Fig. 133 illustrates Stevin's method of marking the given parts of a triangle in the solution of triangles; the K and G represent the Dutch words "klein" and "groot" and signify that the angles are respectively smaller or larger than a right angle.

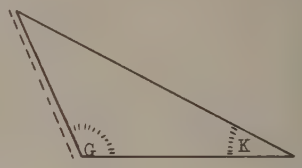


Fig. 133

As a general rule the propositions of the Hypomnemata mathematica are proved separately for right, acute, and obtuse-angled triangles. Each statement in a proof is called a "member." The proofs are more or less similar to those of preceding trigonometric works. The proof of the Cosine Theorem for spherical triangles is identical with the one given by Lansberg.[165] Stevin mentions that his treatment of Vieta's "reciprocal figure" is similar to that of Lansberg.

Stevin's tables of tangents and secants have the respective titles "Canon tangentium Vietae prosinuum" and "Canon secantium Fr. Vietae transsinuosarum." They are computed with a radius equal to 1000000000 for an interval of a minute.

A section of Stevin's trigonometry is entitled "Additamentum de planis multangulis." It is concerned with the trigonometric solution of figures with more than three angles. Fig. 134 is taken from this section of the text.

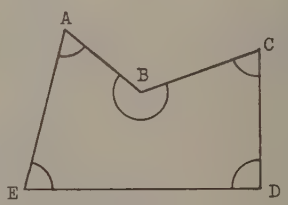


Fig. 134

The work of Pitiscus and his contemporaries concludes the prelogarithmic era of trigonometry. The use of logarithms in the computation of trigonometric tables and the incorporation of trigonometry into analysis are the major points in the subsequent history of trigonometry.

165. See pp. 96-97.

①	25 grad.	26 grad.	27 grad.	28 grad.	29 grad.
30	11079289	11174006	11273820	11378933	11489353
31	11080827	11175627	11275529	11380731	11491445
32	11082366	11177249	11277238	11382530	11469338
33	11083906	11178873	11278949	11384331	11495233
34	11085448	11180499	11280661	11386134	11497140
35	11086990	11182125	11282374	11387938	11495028
36	11088536	11183753	11284089	11389744	11500928
37	11090082	11185383	11285805	11391551	11502829
38	11091629	11187014	11287524	11393359	11504731
39	11093178	11188647	11289244	11395169	11506626
40	11094729	11190281	11290965	11396981	11508532
41	11096280	11191916	11292688	11398793	11510450
42	11097833	11193553	11294412	11400609	11512360
43	11099387	11195191	11296132	11401425	11514271
44	11100943	11196831	11297864	11404243	11516183
45	11102500	11198472	11299593	11406063	11518097
46	11104058	11200114	11301324	11407884	11520013
47	11105618	11201758	11303056	11409706	11521930
48	11107179	11203404	11304789	11411530	11523849
49	11108741	11205051	11306523	11413356	11525770
50	11110306	11206700	11308259	11415183	11527692
51	11111871	11208350	11309996	11417012	11529616
52	11113438	11210001	11311735	11418842	11531542
53	11115006	11211654	11313476	11420673	11533469
54	11116575	11213308	11315218	11422507	11535398
55	11118145	11214963	11316961	11424342	11537328
56	11119717	11216620	11318706	11426178	11539260
57	11121290	11218278	11320452	11428016	11541193
58	11122865	11219938	11322199	11429856	11543128
59	11124442	11221599	11323949	11431689	11545065

tis obviam fore, qui generalem methodum à 3 exempli sequantur, idcoque hoc documentum nostrum supervacuum esse: Ea responsio nolumus, id tum demum verum esse cum perpendicularis demittitur ab ignorato angulo qui lateri 10000 adjacet, quemadmodum hic ducta fuit, non autem cum à reliquo ignorato ducitur: Atque ideo compendium hoc adnotasse erat operæ pretium.

DEMONSTRATIO.

PRAGMATIA investigationis anguli A primi exempli sic demonstratur: In diagrammate 1 definitionis 1 libri tractatus triangulorum, manifestum est ABI esse triangulum rectangulum, hujus latus AE, cum sit æquale sinui recti AE, poterit assumi tanquam sinus recti, undè liquet istiusmodi sinum rectum ejusque tangentem, esse duo crura trianguli rectanguli: Et ex reciprocatione illius, duo quælibet recti anguli crura statui posse alterum pro sinu recti, reliquum verò pro tangente ejusdem: Si itaque in diagraphe 1 exempli hujus propositionis AB assumatur instar sinus recti, BC erit tangens ejus: Verum quandoquidem numeri 16 & 12, non sint ii ad quos canones constituti sunt, iis autem duntaxat proportionales, sic ratiocinabor: Posita AB 16, BC valebit 12, Quanta igitur erit BC si AB (ut in canonibus) sumatur 10000? cócludetur pro BC in canone tangentium 7500. Et arcus ejus, qui deprehenditur 36 gr. 52 ①, erit amplitudo anguli A, ex adverso tangentis BC. Cumque dicta pragmatia istu fundamento innitatur, 36 gr. 52 ①, erit veræ magnitudo anguli A. Similiter demonstrabitur investigatio anguli C. Reliquorum autem demonstratio, ex ipso opere deducitur, & maximam partem è 1 propositione derivatur. CONCLVSIO. Quare si in triangulo plano duo latera cognitum angulum comprehendentia cognita sint: Reliquum latus, reliquosque angulos invenimus. Quod erat faciendum.

5 PROBLEMA. 7 PROPOSITIO.

COGNITIS tribus trianguli plani angulis: trium laterum rationem invenire.

TRES cogniti termini ut generatim exprimam istiusmodi sunt.

DATVM. Esto trianguli plani ABC, angulus A 75 gr. 45 ①, B 67 gr. 23 ①, C 36 gr. 52 ①.

QVAESITVM. Trium laterum rationes inveniendæ sunto.

CONSTRVCTIO.

Sinus anguli A, est pro opposito latere BC 9692.
Sinus anguli B, est pro opposito latere AC 9231.
Sinus anguli C, est pro opposito latere AB 6006.
Quare ut 9692 ad 9231, ita BC ad AC, & sic in cæteris.

DEMONSTRATIO liquidò constat è 1 propositione hujus.

CON-

SPHÆRICA TRIANGVLA TERTIÆ. DISTINCTIONIS

ubi nec angulus rectus, nec latus 90 grad. ut cognita dantur.

Pag. 268 propositione 39. Verum ante inspicito introductionem pag. 269, ut inde quod exemplum sequendum sit cognoscas.

Pag. 281, propositione 40.

Pag. 284, propositione 41.

Pag. 288, propositione 42.

Pag. 292, propositione 43.

Pag. 294, 1 exemplum 44 propositionis.

Pag. 294, 1 exemplum 44 propos.

Pag. 295, 2 exemplum 44 propos.

Pag. 295, 2 exemplum 44 propos.

Plate 40. Pages from Simon Stevin, Hypomnemata mathematica, Leyden, 1608.

Chapter V

THE SUMMARY

An account of the history of trigonometry prior to the Renaissance emphasizes the capabilities of the Moslem mind. Arabic civilization was heir to Hellenistic civilization, which it interpreted and in many respects elaborated, as witnessed by the first independent treatment of trigonometry, the Treatise on the Quadrilateral by Naṣir ad-Dīn at-Tūsī, wherein the ancient theory of the quadrilateral reached its highest development. Here the Hindu half chord replaced the whole chord of the Greeks, the Sine Law for a plane triangle with its first proof by Al-Bīrūnī was given, and the tangent functions was employed.

The study of the works of the learned Moslems in Spain and the revival of interest in the classical authors revealed a world of scientific knowledge not known to medieval Europe and stimulated a search for further information. In trigonometry, however, the work of the Spaniards lagged behind that of the Eastern Arabs.

The rebirth of trigonometry in Europe was sponsored chiefly by Regiomontanus, the pupil of Peurbach who was the first to construct a table of sines with the radius 600 000 for intervals of ten minutes. Regiomontanus sought to do for the West what Naṣir ad-Dīn had done two centuries earlier for the East, namely, construct a uniform foundation and systematic ordering of trigonometric knowledge as is evidenced by his De triangulis libri quinque. Here, for the first time, the trigonometric formula for the area of a triangle was implied, and the Law of Cosines for spherical triangles was put into a form serviceable for calculation. The tangent function was not included; in this way and in other ways the trigonometry of Regiomontanus was not as advanced as that of Naṣir ad-Dīn and other writers of Persia and Arabia.

Regiomontanus also endeavored to improve the tables for trigonometric computations by the construction of a table of sines first for the radius equal to 6×10^6 and then for it equal to 10^7. The Tabulae directionum, which contains the "tabula fecunda," proves that Regiomontanus finally knew and appreciated the tangent function.

Johann Werner and Peter Apian, contemporaries of Regiomontanus, did some work in trigonometry. De triangulis per maximorum circulorum segmenta constructis libri V, the trigonometric treatise of the former remained in manuscript form until 1907, although its publication was begun by Rheticus in 1557. It shows no particular advance over the trigonometry of Regiomontanus. The fame of Peter Apian rests upon his astronomical discoveries and not on any original work in mathematics. However, the terminology he established in his Instrumentum sinuum seu primii mobilis of 1541 was adopted by Maurolyco in his treatise of 1558.

The trigonometric portion of the De revolutionibus orbium coelestium of Copernicus, published separately by Rheticus, reveals a striking similarity with the trigonometric work of Regiomontanus. The conclusion has been reached that Copernicus received information concerning the trigonometry of Regiomontanus from Rheticus. The reasons for drawing this conclusion are taken from three sources. The first is an extract from the Commentariolus wherein Copernicus expressly states that he is withholding all mathematical demonstrations from this work and reserving them for his larger astronomical treatise, thereby revealing nothing as to the status of his trigonometry previous to the publication of the De lateribus et angulis triangulorum. The second source is the dedicatory epistle of the De lateribus et angulis triangulorum. Here Rheticus asserts that, prior to his visit to Ermland, the trigonometry of Regiomontanus was not available to Copernicus. Valentine Otho's "Ad Candidum Leotorem" in Opus palatinum de

triangulis, the posthumous work of Rheticus, is the third source. Here Otho maintains that Rheticus had thoroughly mastered the trigonometry of Regiomontanus before he went to Ermland. Otho further asserts that Copernicus had imitated his predecessor, Ptolemy, who borrowed from Menelaus and Theodosius, by taking what was essential to his own work from Ptolemy and Regiomontanus. Otho also mentions that when Copernicus could find no suitable mathematical demonstrations for the doctrine of the first motion from the books which were "only too few" in his own library, it was the constant urging of Rheticus and the high regard Copernicus had for him which induced Copernicus to continue his work until he found the desired demonstrations.

In their treatises on triangles Bressieu, Clavius, and Stevin indicate several errors in the trigonometric theorems of Copernicus. These errors, or more precisely incomplete proofs, do not occur in the work of Regiomontanus, hence they must have originated with Copernicus. In condensing the work of Regiomontanus, the importance of stating the possibility of having more than one solution to several of the cases in the solution of spherical triangles must not have occurred to Copernicus.

After Rheticus resigned his position as professor of mathematics at the University of Leipzig, he devoted his time to the calculation of trigonometric tables. He computed a table of sines, tangents, secants, as well as their complementary functions with the radius equal to 10^7 for the interval of ten seconds. These tables appeared in 1551 under the title Canon doctrinae triangulorum. In 1579 Vieta, in his Canon mathematicus seu ad triangula cum appendicibus, which also contains some trigonometry, extended these tables, giving the values to seven places of all six trigonometric functions for every second arc. Rheticus began the construction of tables of tangents and secants for the radius equal to 10^{15} for the inverval of ten seconds, but died before finishing them. Valentine Otho, a faithful pupil, completed the tables in 1596; they were printed as the Opus palatinum de triangulis, begun by George Joachim Rheticus and completed by L. Valentine Otho. The Thesaurus mathematicus, a table of sines, was also begun by Rheticus and completed and published after his death by Valentine Otho in 1598. Pitiscus republished it in 1613.

Although Rheticus has won an independent place in mathematics because of the tables he constructed, his views on trigonometrical lines and his development of plane trigonometry show that he was more than a ready calculator. He discarded the traditional consideration of trigonometric functions with respect to an arc of a circle by constructing a right triangle and making the functions depend directly on its angles. Rheticus conceived the idea of a table of secants from the calculation of the hypotenuse of a right triangle.

Maurolyco was a close student of the ancients and adopted their methods. He imitated his immediate predecessors and contemporaries and either reproduced or extended their work. His trigonometry shows the influence of Geber, Regiomontanus, and Rheticus; his notation for the trigonometric functions can be traced to Peter Apian. Maurolyco wrote two treatises on trigonometry, the first of which, published in 1558, contains trigonometric relations between sides and angles of spherical triangles, but these formulas are not applied to the formal solution of spherical triangles. His second treatise is embodied in the first book of "Geometricarum Quaestionum," written in 1555 but not published until 1876. Here Maurolyco introduces the secant function and remarks that his "tabula benefica" (secants) is not less useful than the "tabula fecunda" of Regiomontanus. Arguments have been advanced for and against the possibility that Maurolyco took his table of secants from Rheticus; the priority of Rheticus, however, is not disputed.

Besides his contributions to pre-logarithmic computation, Vieta advanced the science of trigonometry in the manipulation of trigonometric formulas by algebraic methods. By employing a systematic method in the application of algebra to trigonometry he obtained many of the fundamental identities of plane trigonometry algebraically, notably the expressions for multiple angles. Vieta was the first to use the method of polar triangles.

Vieta's innovations in trigonometry are somewhat comparable with his invention of the literal symbolism which he introduced for a general unknown angle into his trigonometric treatise of 1579 and which was the "initial and crucial step toward a new mathematics."

The Metricis astronomiae of Maurice Bressieu, published in 1581, two years after the appearance of Vieta's Canon mathematicus, shows that Bressieu was familiar with the tables of Rheticus. A minor innovation, possibly original with Bressieu, is that of marking the given parts of a figure in the solution of triangles. His tables, computed for sines, tangents, secants, and their cofunctions, resemble the sixth table of the Canon mathematicus of Vieta; he used the radius 1 divided into 60 parts, each part into 60 primes and each prime into 60 seconds, thus sin 45° equals $42^{\circ}25'35''$.

Although Bressieu made no exceptional advances in trigonometry and appears to have borrowed much from his predecessors, he shows considerable independence in giving alternate methods of proof.

In his Geometria rotundi of 1583 Thomas Finck introduced the use of the geometric terms "tangent" and "secant" for the trigonometric functions that still bear these names. Finck did not define the trigonometric functions as parts of a right triangle as Rheticus and Vieta had done, but considered the trigonometric lines as lines of a circle. He computed his tables in a manner similar to the method of Bressieu, but neglected to arrange them for the reading of the complementary functions. His treatise contains the statement and proof of the Law of Tangents, and is, in all probability, the first to do so. Vieta gave this formula in a more elegant form ten years later.

In his spherical trigonometry Finck followed Bressieu; in one thing, however, he advanced beyond Bressieu, that is, in the statement of the Cosine Theorem, the form of which he took from Regiomontanus and applied it for the first time to the solution of an oblique spherical triangle when two sides and the included angle are given.

The Mathematical Jewel of Blagrave published in 1585 did little or nothing for the advancement of trigonometry. The author's purpose was to replace by means of the planisphere the method of solving triangles by trigonometric calculations. Blagrave's work shows his familiarity with the treatises of Regiomontanus, Copernicus and Apian, but exhibits an unfamiliarity with the works of contemporary authors.

The trigonometric treatises of Christopher Clavius are of much lesser merit than those of Vieta in so far as they lack the originality that characterized Vieta's work. Clavius made no real technical advance in trigonometry; his work served to transmit to his successors the best that was given by previous writers. Romanus dedicated his Ideae mathematicae pars prima to Clavius; in his Canon triangulorum he advanced beyond Clavius when he reduced the twenty-eight formulas for right-angled spherical triangles to six.

There is a certain amount of similarity between the Triangulorum libri quattuor of Lansberg which was published in 1591 and the treatment given to trigonometry by Stevin in 1608. Stevin, for instance, states almost verbatim Lansberg's proof for the Cosine Theorem; he mentions that his description of Vieta's "reciprocal figure" is the modified form given by Lansberg. He probably took the idea of marking the known parts of a triangle from Bressieu, but used his own device.

If the trigonometry embodied in the Diclides coelometricae (1602) of Torporley, the author of the most unique book ever written on trigonometry, had not been disguised under so many metaphors, credit for the invention of "circular parts" might now be ascribed to him instead of to Napier. Further, the first of his two tables was taken over by Magini in his monumental work on spherical trigonometry.

The three trigonometric treatises of Magini are De planis triangulis of 1592, the Primum mobile and Tabulae primum mobile, both published in 1604. The first treatise was not as advanced as the trigonometries of previous writers. However, the spherical trigonometry in the Primum mobile shows Magini's complete mastery of the subject.

Magini restates the six forms of right-angles spherical triangles given by Rheticus and supplements them by his own classification of oblique triangles under ten forms. His treatment of polar triangles is a modification of Vieta's method.

For clarity of ideas and simplicity of form the Trigonometriae of Pitiscus in the editions of 1600-1612, is the most outstanding treatise of trigonometry developed before the introduction of logarithms. From an extant copy of some lecture notes entitled "Lectiones in trigonometriam (Bartholomaei) Pitisci" of Melchior Jöstel, it is apparent that this treatise served as a source for the lectures that were given in trigonometry in the universities of Germany at the end of the sixteenth century. The fact that the Trigonometriae of Pitiscus was reëdited three times and translated into English (1614) so soon after its appearance on the Continent indicates its place in contemporary mathematics.

For purposes of reference, a summary of the terminology and notations of the respective authors treated in this study will be given in tabular form. An asterisk beside an author's work indicates that the author's nomenclature and notation used therein have been taken solely from secondary sources which are:

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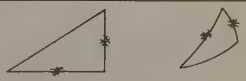
	VERS SINE	RADIUS	MISCELLANEOUS NOTES
Regio	us versus	sinus totus	"Fecunda" is the name of the table of tangents. Regiomontanus uses the term "casus" for the segments of the base made by a perpendicular drawn from the opposite vertex.
Apian	us versus itta	sinus totus, sinus maximus, perfectus	
Coper		dimedia diametri	The symbols for minutes and seconds were probably used by Rheticus and not by Copernicus.
Rheti		Subtendens angulum rectum	The (1), (2), (3) refer to the triangles of the first, second, and third series respectively.
Maurc	itta	sinus totus, sinus maximus, radius, semidiameter	"Benfica" is the name of the table of secants.
		sinus totus	The term "casus" is also used by Maurolyco to designate the portions of the base made by a perpendicular drawn from the opposite vertex.
Vieta		sinus totus, sinus hypotheticus	Vieta wrote the functions and complementary functions of an angle in the following way: Sinus A, Sinus Ā; Fecunda A, Fecunda Ā; Hypotenusa A, Hypotenusa Ā.
Bress		radius, semidiameter	Bressieu was the first to use a device for marking the given sides of a triangle. 
Finck	us secundus, . sec.		
Blagi	plement of Syne	Syne totall	
Clavi	us versus	sinus totus	

	VERS SINE	RADIUS	MISCELLANEOUS NOTES
		sinus totus	
	sinus versus	radius	Lansberg also had a device for marking the given parts of a triangle.
	sinus versus	radius, sinus totus, sinus integer	
	sinus totus	sinus totus	Vieta considered triangles of the first, second, and third series.
	sinus versus, shaft	sinus totus, total sine	
		sinus totus	The (1), (2), (3) refer to the triangles of the first, second, and third series.
		sinus totus, S.T.	In the solution of triangles, Jöstel marked the given parts with a stroke and the required parts with a circle.
	sinus versus	radius	In plane triangles Pitiscus marked the given parts; in spherical triangles, the given and required. The 1600 edition is not the first; a much smaller work under the same title was published in 1595.
		sinus totus	
		sinus totus	
	sinus-sagitta sagitta, sagitta sinus	semidiameter	Stevin also had a device for marking the given parts of a plane triangle. The K and G represent the Dutch words klein and groot and designate an acute and obtuse angle respectively.
	sinus versus	radius, sinus, totus, sinus integer	

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